TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 352, Number 7, Pages 3045–3064
S 0002-9947(99)02429-0
Article electronically published on May 21, 1999

PROJECTIVE MANIFOLDS WITH SMALL PLURIDEGREES

MAURO C. BELTRAMETTI AND ANDREW J. SOMMESE

Abstract. Let $\mathcal{L}$ be a very ample line bundle on a connected complex projective manifold $\mathcal{M}$ of dimension $n \geq 3$. Except for a short list of degenerate pairs $(\mathcal{M}, \mathcal{L})$, $\kappa(K_{\mathcal{M}} + (n-2)\mathcal{L}) = n$ and there exists a morphism $\pi : \mathcal{M} \to M$ expressing $\mathcal{M}$ as the blowup of a projective manifold M at a finite set B, with $\mathcal{K}_M := K_M + (n-2)L$ nef and big for the ample line bundle $L := (\pi_*\mathcal{L})^{**}$. The projective geometry of $(\mathcal{M}, \mathcal{L})$ is largely controlled by the pluridegrees $d_j := L^{n-j} \cdot (K_M + (n-2)L)^j$ for $j = 0, \ldots, n$, of $(\mathcal{M}, \mathcal{L})$. For example, $d_0 + d_1 = 2g - 2$, where g is the genus of a curve section of $(\mathcal{M}, \mathcal{L})$, and d_2 is equal to the self-intersection of the canonical divisor of the minimal model of a surface section of $(\mathcal{M}, \mathcal{L})$. In this article, a detailed analysis is made of the pluridegrees of $(\mathcal{M}, \mathcal{L})$. The restrictions found are used to give a new lower bound for the dimension of the space of sections of $\mathcal{K}_M$. The inequalities for the pluridegrees, that are presented in this article, will be used in a sequel to study the sheet number of the morphism associated to $|2(K_{\mathcal{M}} + (n-2)\mathcal{L})|$.

Introduction

Let $\mathcal{L}$ be a very ample line bundle on a connected complex projective manifold $\mathcal{M}$ of dimension $n \geq 3$. Except for a short list of degenerate pairs $(\mathcal{M}, \mathcal{L})$, it follows that $\kappa(K_{\mathcal{M}} + (n-2)\mathcal{L}) = n$ and that there exists a morphism $\pi : \mathcal{M} \to M$ expressing $\mathcal{M}$ as the blowup of a projective manifold M at a finite set B, with $\mathcal{K}_M := K_M + (n-2)L$ nef and big for the ample line bundle $L := (\pi_*\mathcal{L})^{**}$. Note that if $\mathcal{K}_M$ is nef and big, then a smooth element $S \in |\mathcal{L}|$ is of general type and the map onto the minimal model of S is given by $\pi_S : S \to \pi(S)$. Moreover π_S expresses S as the blowup of $\pi(S)$ on a finite set of cardinality γ. The projective geometry of $(\mathcal{M}, \mathcal{L})$ is largely controlled by the pluridegrees $d_j := L^{n-j} \cdot (K_M + (n-2)L)^j$, for $j = 0, \ldots, n$, of $(\mathcal{M}, \mathcal{L})$. The usual invariants that come up in classification are expressed simply in terms of these invariants; e.g., see [7, 3]. For example, $d_0 = \widehat{d} + \gamma$ where $\widehat{d}$ denotes the degree of $\mathcal{M}$ relative to the embedding given by $|\mathcal{L}|$, $d_0 + d_1 = 2g - 2$ where g is the genus of a curve section of $(\mathcal{M}, \mathcal{L})$, and d_2 is equal to the self-intersection of the canonical divisor of the minimal model of a surface section of $(\mathcal{M}, \mathcal{L})$. In this article, a detailed analysis is made of the pluridegrees of $(\mathcal{M}, \mathcal{L})$.

Here is a detailed summary of the results of this article. Theorem (2.1) details the exceptions to inequalities of the form $d_j \geq d_{j-1} + 2$ in dimensions ≥ 5, and

Received by the editors February 8, 1998.

1991 *Mathematics Subject Classification.* Primary 14J40; Secondary 14M99, 14C20.

Key words and phrases. Smooth complex polarized n-fold, very ample line bundle, adjunction theory, log-general type, pluridegrees.

Theorem (2.2) details the exceptions to inequalities of the form $2d_j \geq d_{j-1}$ in dimension 4.

Consider for example Theorem (3.1) in the case of dimensions ≥ 6. Adjunction theory [7] shows that the Kodaira dimension of $K_{\mathcal{M}} + (n-3)\mathcal{L}$ is n except for a short list of exceptional varieties (scrolls, quadric fibrations, and the like). In the case that the Kodaira dimension of $K_{\mathcal{M}} + (n-3)\mathcal{L}$ is n, we have a first reduction (M, L) with $K_M + (n-2)L$ nef and big, and a second reduction $(X, \mathcal{D})$, $\varphi : M \to X$, with $K_X + (n-3)\mathcal{K} \approx (n-2)(K_X + (n-3)\mathcal{D})$ nef and big, where $\mathcal{K}$ is an ample line bundle on X such that $K_M + (n-2)L$ is the pullback of $\mathcal{K}$ under φ (see (1.1)). Theorem (3.1) asserts that in this situation, either we have the inequalities $d_j \geq 2\left(\frac{n-2}{n+1}\right) d_{j-1}$ for $j = 1, \ldots, n$, or we are in a very restricted situation, e.g., either $\mathcal{M}$ is a special degree 364 sixfold (which might not exist), or $\mathcal{M}$ is a special sort of blowup $\sigma : \mathcal{M} \to \mathbb{P}^8$ with $(\sigma_*\mathcal{L})^{**} = \mathcal{O}_{\mathbb{P}^8}(2)$, e.g., $\mathcal{M}$ is the blowup of $\mathbb{P}^8$ along a line $\ell \subset \mathbb{P}^8$ and $\mathcal{L} := \sigma^*\mathcal{O}_{\mathbb{P}^8}(2) - \sigma^{-1}(\ell)$. The inequalities $d_j \geq 2\left(\frac{n-2}{n+1}\right) d_{j-1}$ for $j = 1, \ldots, n$ are very strong and get stronger as n increases. For example, in dimension six, $d_j \geq \frac{8}{7} d_{j-1}$ for $j = 1, \ldots, n$. Among other things this implies that the genus g of a curve section C of $\mathcal{M}$ is strictly bounded below by $\frac{8}{7}\widehat{d}$ where $\widehat{d}$ denotes the degree of $\mathcal{M}$ relative to the embedding given by $|\mathcal{L}|$. It also implies that $K^2_{\pi(S)}$ for the minimal model $\pi(S)$ of a smooth surface section S of $(\mathcal{M}, \mathcal{L})$ is strictly bounded below by $\frac{64}{49}\widehat{d}$. In applications, results such as this are very effective in ruling out small invariant curves and surfaces with exceptional behavior.

Results that hold in the more difficult dimensions, 3, 4, and 5, are presented in Theorems (4.1), (5.1), (6.2), and (7.1). As a first application of our results we give in Proposition (8.1) a new lower bound for the dimension of the space of sections of $\mathcal{K}_M$ when $n \geq 5$. This result says that if the Kodaira dimension of $K_{\mathcal{M}} + (n-3)\mathcal{L}$ is n, then $h^0(K_{\mathcal{M}} + (n-2)\mathcal{L}) \geq 5$. This implies in particular that a smooth surface section of $\mathcal{M}$ relative to the embedding given by $\mathcal{L}$ possesses at least 5 linearly independent holomorphic two forms.

In [9] we used the results of this article to give sharp bounds for the degree of the morphism associated to $|2(K_{\mathcal{M}} + (n-2)\mathcal{L})|$.

Both authors would like to thank the Duncan Chair of the University of Notre Dame for the research support it provided in March of 1997. The first author would like to thank the Max-Planck-Institut für Mathematik in Bonn, where the final parts of this work, including the preparation of the manuscript, were done in July 1997. The second author would like to thank the Alexander von Humboldt Stiftung for their support in July 1997.

1. Background material

We work over the complex field $\mathbb{C}$. The notations used in this paper are standard from algebraic geometry. In particular we follow the notation of [8, (1.1)]. We define $\kappa(D)$, the Kodaira dimension of a $\mathbb{Q}$-Cartier divisor, D, on a projective variety V, to be $\kappa(ND)$, where N is a positive integer such that ND is Cartier; $\kappa(V) := \kappa(K_V)$, the Kodaira dimension of V, for V smooth. We refer to [8, §1] for a summary of a number of results we use extensively through this paper, e.g., the genus formula [8, (1.2)], various double point formulae [8, Prop. (1.5), Lemma (1.6), Cor. (1.7)], Tsuji's inequality [8, Prop. (1.8)], and Castelnuovo's inequality [8, (1.9)].

1.1. **Reductions.** (See e.g., [7, Chapters 7, 12].) Let $(\mathcal{M}, \mathcal{L})$ be a smooth variety of dimension $n \geq 2$ polarized with a very ample line bundle $\mathcal{L}$. Except for an explicit list of well understood pairs $(\mathcal{M}, \mathcal{L})$ (see in particular [7, §§7.2, 7.3, 7.4]) it follows that:

a) $K_{\mathcal{M}} + (n-1)\mathcal{L}$ is spanned and big, and there is a morphism $\pi : \mathcal{M} \to M$ expressing $\mathcal{M}$ as the blowing up of a projective manifold M at a finite set of points, B, such that $L := (\pi_*\mathcal{L})^{**}$ is ample and $\mathcal{L} \approx \pi^*L - [\pi^{-1}(B)]$ or, equivalently, $K_{\mathcal{M}}+(n-1)\mathcal{L} \approx \pi^*(K_M+(n-1)L)$. Furthermore $K_M+(n-1)L$ is very ample. The pair (M, L), which is unique up to isomorphism, is called *the first reduction* of $(\mathcal{M}, \mathcal{L})$.

b) $K_M + (n-2)L$ is nef and big, for $n \geq 3$.

If $K_M + (n-2)L$ is nef and big, then there is a morphism $\varphi : M \to X$ with connected fibers and normal image and an ample line bundle $\mathcal{K}$ on X such that $K_M+(n-2)L \approx \varphi^*\mathcal{K}$. The morphism φ is very well behaved, e.g., X has terminal, 2-Gorenstein (i.e., $2K_X$ is a line bundle) isolated singularities and $\mathcal{K} \approx K_X+(n-2)\mathcal{D}$, where $\mathcal{D} := (\varphi_*L)^{**}$ is a 2-Cartier divisor such that $2L \approx \varphi^*(2\mathcal{D}) - \Delta$ for some effective divisor Δ on M which is φ-exceptional and $\dim \varphi(\Delta) \leq 1$ (see [7, (7.5.7)]). The pair $(X, \mathcal{D})$ is known as the *second reduction* of $(\mathcal{M}, \mathcal{L})$. Notice that from $K_M + (n-2)L \approx \varphi^*(K_X + (n-2)\mathcal{D})$ and $2L \approx \varphi^*(2\mathcal{D}) - \Delta$ it follows that $2K_M \approx \varphi^*(2K_X) + (n-2)\Delta$. For further details and definitions we refer to [7, §§7.5, 7.6, 7.7 and Chapter 12].

If Y is an irreducible subvariety of X we will denote by $\mathcal{D}_Y$ the double dual sheaf $(\mathcal{D}_{|Y})^{**}$, where $\mathcal{D}_{|Y}$ is the restriction of $\mathcal{D}$ to Y as a sheaf. Moreover, for any k-dimensional irreducible subvariety $Y \subset X$, we set

$$\mathcal{D}^k \cdot Y := \frac{(2\mathcal{D})^k \cdot Y}{2^k}.$$

Most of the results of this paper hold under the assumption that $\kappa(K_{\mathcal{M}}+(n-2)\mathcal{L}) = n$. Notice that this is equivalent to say that $K_M + (n-2)L$ is nef and big (see [7, (7.6.9)]). Note also that this is equivalent to the usual definition of log-general type for threefolds, and is usually taken in the adjunction theory literature (see e.g., [7]) as the definition of log-general type in dimension greater than three. Let $\widehat{S}$ be the smooth surface obtained as transversal intersection of $n-2$ general members of $|\mathcal{L}|$ and let $S := \pi(\widehat{S})$ be the corresponding smooth surface in M. Since $K_M+(n-2)L$ is nef and big, the canonical bundle K_S of S is nef and big, so that S is a minimal surface of general type (see also [7, (7.6.10)]). Note that we have the strict Miyaoka inequality $K_S \cdot K_S < 9\chi(\mathcal{O}_S)$ (see [8, Eq. (4)]).

For further properties of polarized pairs $(\mathcal{M}, \mathcal{L})$ with $\kappa(K_{\mathcal{M}}+(n-2)\mathcal{L}) = n$ see e.g., [7, §13.2] and [6, (0.10)].

We use, throughout the paper, the following results from adjunction theory. We refer to [7, (7.7.9)] for the first one. The second one is essentially due to Fujita [10] and also follows from the results of §2 of [5].

Proposition 1.2. *Let $\mathcal{M}$ be an n-dimensional, smooth, connected variety and $\mathcal{L}$ an ample line bundle on $\mathcal{M}$. Assume $n \geq 4$. Then $\kappa(K_{\mathcal{M}} + (n-3)\mathcal{L}) = n$ if and only if the second reduction, $(X, \mathcal{D})$, of $(\mathcal{M}, \mathcal{L})$ exists, and either $K_X + (n-3)\mathcal{D}$ or equivalently $K_X + (n-3)\mathcal{K}$ is nef and big, where $\mathcal{K} \approx K_X + (n-2)\mathcal{D}$.*

Proposition 1.3. *Let X be an n-dimensional, irreducible normal variety with terminal singularities and let H be an ample line bundle on X. Assume that X is*

$\mathbb{Q}$-factorial and $n \geq 2$. Then $K_X + (n-1)H$ is nef (respectively nef and big) if and only if $\kappa(K_X + (n-1)H) \geq 0$ (respectively $\kappa(K_X + (n-1)H) = n$).

1.4. **Pluridegrees.** For more details on the pluridegrees we refer to [7, Chap. 13]. Let $(\mathcal{M}, \mathcal{L})$, (M, L) be as in (1.1). Define the *pluridegrees*, for $j = 0, \ldots, n = \dim \mathcal{M}$, by $\widehat{d}_j := (K_{\mathcal{M}} + (n-2)\mathcal{L})^j \cdot \mathcal{L}^{n-j}$ and $d_j := (K_M + (n-2)L)^j \cdot L^{n-j}$. If γ denotes the number of points blown up under $\pi : \mathcal{M} \to M$, then $\widehat{d}_j = d_j - (-1)^j \gamma$. We put $\widehat{d} := \widehat{d}_0$, $d := d_0$. We will repeatedly use the Hodge index inequalities (see [8, Eq. (2)] and [7, (2.5.1), (13.1)]) and the parity relations (see [8, Eq. (3)] and [7, Lemma (13.1.1)]) for these invariants. Moreover if $K_M + (n-2)L$ is nef and big, i.e., if $\kappa(K_{\mathcal{M}} + (n-2)\mathcal{L}) = n$, the d_j's are positive.

If the second reduction, $(X, \mathcal{D})$, $\varphi : M \to X$, with $\mathcal{D} = (\varphi_* L)^{**}$, $\mathcal{K} \approx K_X + (n-2)\mathcal{D}$, of $(\mathcal{M}, \mathcal{L})$ exists, then we define $d'_j := \mathcal{K}^j \cdot \mathcal{D}^{n-j}$, $j = 0, \ldots, n$, $d' := d'_0$. Note that

$$d_j = d'_j \text{ for } j \geq 2. \tag{1}$$

To see this recall that $2L \approx \varphi^*(2\mathcal{D}) - \Delta$ for some effective Cartier divisor Δ which is φ-exceptional (see (1.1)) and compute, for $j \geq 2$,

$$\begin{aligned} 2^{n-j} d_j &= (K_M + (n-2)L)^j \cdot (2L)^{n-j} \\ &= (\varphi^* \mathcal{K})^j \cdot (\varphi^*(2\mathcal{D}) - \Delta)^{n-j} = 2^{n-j} \mathcal{K}^j \cdot \mathcal{D}^{n-j} = 2^{n-j} d'_j, \end{aligned}$$

where the last equality follows from the fact that $\dim \varphi(\Delta) \leq 1$.

We need the following lower bound for the degree.

Lemma 1.5. *Let $\mathcal{M}$ be a smooth n-fold polarized by a very ample line bundle $\mathcal{L}$. Assume that $\kappa(K_{\mathcal{M}} + (n-2)\mathcal{L}) = n$. Let (M, L) be the first reduction of $(\mathcal{M}, \mathcal{L})$. Then either $|\mathcal{L}|$ embeds $\mathcal{M}$ in $\mathbb{P}^{n+1}$ or $d \geq \widehat{d} := \mathcal{L}^n \geq 8$.*

Proof. We can assume that $\Gamma(\mathcal{L})$ embeds $\mathcal{M}$ in $\mathbb{P}^N$ with $N \geq n+2$. Let $\widehat{S}$ be the smooth surface obtained as the transversal intersection of $n-2$ general members of $|\mathcal{L}|$. Since $\widehat{S}$ is of general type, we have by a result of Castelnuovo (see [7, (8.1)], [13, (0.6)]), that $\widehat{d} > 2(N-n) + 2 \geq 6$. Thus $d \geq \widehat{d} \geq 7$. Assume $\widehat{d} = 7$. Then $N = n+2$ since otherwise $\widehat{d} > 2(N-n) + 2 \geq 8$. Thus by Castelnuovo's bound we conclude that $g(C) \leq 6$ for any smooth curve section C of $\mathcal{M} \subset \mathbb{P}^{n+2}$. Therefore, by the genus formula, $7 + d_1 \leq 7 + \widehat{d}_1 = 2g(C) - 2 \leq 10$, or $d_1 \leq 3$. But then $d_2 \leq 1$ by the Hodge index relation $d_1^2 \geq d d_2$. Since $\kappa(K_{\mathcal{M}} + (n-2)\mathcal{L}) = n$ we know that $d_j \geq 1$ for each $j = 1, \ldots, n$. Therefore we conclude that $d_2 = 1$, and using $d_2^2 \geq d_1 d_3$ that $d_1 = 1$. This gives the absurdity that $d = 1$. Q.E.D.

We need the following useful consequence of the Levi extension theorem.

Lemma 1.6. *Let L be a line bundle on an irreducible projective variety V and let $p : V \to Y$ be a birational surjective morphism with Y normal. Then we have an inclusion*

$$H^0(V, L) \subset H^0(Y, (p_* L)^{**}).$$

2. Exceptions to nefness and bigness

Let $\mathcal{M}$ be a smooth connected n-fold polarized by a very ample line bundle $\mathcal{L}$. Assume that $\kappa(K_{\mathcal{M}} + (n-2)\mathcal{L}) = n$. Let $\pi : (\mathcal{M}, \mathcal{L}) \to (M, L)$, $\varphi : (M, L) \to (X, \mathcal{D})$, $\mathcal{K} \approx K_X + (n-2)\mathcal{D}$, be the first and the second reduction of $(\mathcal{M}, \mathcal{L})$ as in (1.1). In this section we study the exceptions to nefness and bigness of $K_X + (n-3)\mathcal{K}$ when $n \geq 5$, and the exceptions to nefness and bigness of $K_X + 3\mathcal{K}$ for $n = 4$.

First assume $n \geq 5$. From [7, (7.7.2), (7.7.3), (7.7.5), (7.7.6)] we know that $K_X + (n-3)\mathcal{K}$ is nef and big unless either

i) $n = 6$, $(X, \mathcal{K}) \cong (\mathbb{P}^6, \mathcal{O}_{\mathbb{P}^6}(1))$; or
ii) $n = 5$, $(X, \mathcal{K}) \cong (\mathcal{Q}, \mathcal{O}_{\mathcal{Q}}(1))$, $\mathcal{Q}$ a quadric in $\mathbb{P}^6$; or
iii) $n = 5$, X is a $\mathbb{P}^4$-bundle over a smooth curve, $\mathcal{K}_F \approx \mathcal{O}_{\mathbb{P}^4}(1)$ for any fiber F; or
iv) $n = 5$, X is a singular 2-Gorenstein Fano 5-fold described in [7, (7.7.5)]; or
v) $(X, \mathcal{K})$ is a scroll over a normal 4-fold Y, $\psi : X \to Y$, $\mathcal{K}_{|F} \approx \mathcal{O}_{\mathbb{P}^4}(1)$ for a general fiber F; or
vi) $(X, \mathcal{K})$ is a quadric fibration, $\psi : X \to Y$, over a normal 3-fold Y; or
vii) $(X, \mathcal{K})$ is a Del Pezzo fibration, $\psi : X \to Y$, over a normal surface Y; or
viii) $(X, \mathcal{K})$ is a Mukai fibration, $\psi : X \to Y$, over a smooth curve Y; or
ix) X is a Fano n-fold with $K_X \approx -(n-3)\mathcal{K}$ if n is even and $2K_X \approx -2(n-3)\mathcal{K}$ if n is odd.

In cases i) and ii) one has $d_n = d_6 = 1$, $d_n = d_5 = 2$ respectively.

In cases v), vi), vii) and viii) we have $K_X + (n-3)\mathcal{K} \approx \psi^*\mathcal{H}$ if n is even and $2(K_X + (n-3)\mathcal{K}) \approx \psi^*\mathcal{H}$ if n is odd, for some ample line bundle $\mathcal{H}$ on Y. In both cases, since $\mathcal{K} \approx K_X + (n-2)\mathcal{D}$, we conclude that $\mathcal{K} - \mathcal{D}$ is numerically effective. In this case, dotting with $\mathcal{K}^{n-1}$, we get the relation $d'_n > d'_{n-1}$ on X, or, recalling (1), $d_n > d_{n-1}$ on M. By using the Hodge index relations we find

$$d_n > d_{n-1} > d_{n-2} > \cdots > d_1 > d. \tag{2}$$

According to whether n is odd or even, we have the following bounds:

$d_k \geq d_{k-1} + 2$ for $1 \leq k \leq n$; thus $d_n \geq d + 2n$ for n odd, and
$d_n \geq d_{n-1} + 1$, $d_k \geq d_{k-1} + 2$ for $1 \leq k \leq n-1$; thus $d_n \geq d + 2n - 1$ for n even.

Assume n is odd. By parity [7, Lemma (13.1.1)] we have $d_n \geq d_{n-1} + 2$. Let t be the biggest integer, $t < n$, such that $d_t < d_{t-1} + 2$. Since $d_t > d_{t-1}$, thus we get $d_t = d_{t-1} + 1$. Therefore $d_{t-1}d_{t+1} \leq d_t^2 = d_t d_{t-1} + d_t$, or $d_{t-1}(d_{t+1} - d_t) \leq d_t$. By the assumption on t, one has $d_{t+1} - d_t \geq 2$. Then we find $2d_{t-1} \leq d_t$, whence $2d_{t-1} \leq d_t = d_{t-1} + 1$. This gives $d_{t-1} = 1$ and hence $d = 1$ by the Hodge index inequalities. Therefore $\widehat{d} = 1$ and $(\mathcal{M}, \mathcal{L}) \cong (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$, contradicting the assumption $\kappa(K_{\mathcal{M}} + (n-2)\mathcal{L}) = n$.

Assume n is even. By parity we have $d_{n-1} \geq d_{n-2} + 2$. Let t be the biggest integer, $t \leq n$, such that $d_t < d_{t-1} + 2$. Since $d_t > d_{t-1}$, then we have $d_t = d_{t-1} + 1$. It may happen that $d_n = d_{n-1} + 1$. If this is not the case, we may assume $t < n$ and exactly the same argument as above leads to a contradiction.

The arguments above show the following result.

Theorem 2.1. *Let $\mathcal{M}$ be a smooth connected n-fold polarized by a very ample line bundle $\mathcal{L}$, with $n \geq 5$. Assume that $\kappa(K_{\mathcal{M}} + (n-2)\mathcal{L}) = n$. Let (M, L), $(X, \mathcal{D})$ be the first and the second reduction of $(\mathcal{M}, \mathcal{L})$ with $\mathcal{K} \approx K_X + (n-2)\mathcal{D}$ as in (1.1). Let d_j, $j = 0, \ldots, n$, $d_0 = d$, be the pluridegrees of (M, L). Assume that $K_X + (n-3)\mathcal{K}$ is not nef and big. Then either*

1. *$d_k \geq d_{k-1} + 2$ for $k = 1, \ldots, n$, and hence $d_n \geq d + 2n$, if n is odd, and $d_n \geq d_{n-1} + 1$, $d_k \geq d_{k-1} + 2$ for $k = 1, \ldots, n-1$, and hence $d_n \geq d + 2n - 1$, if n is even; or*

2. *$n = 6$, $(X, \mathcal{K}) \cong (\mathbb{P}^6, \mathcal{O}_{\mathbb{P}^6}(1))$; or*
3. *$n = 5$, $(X, \mathcal{K}) \cong (\mathcal{Q}, \mathcal{O}_{\mathcal{Q}}(1))$, $\mathcal{Q}$ a quadric in $\mathbb{P}^6$; or*
4. *$n = 5$, X is a $\mathbb{P}^4$-bundle over a smooth curve, $\mathcal{K}_{|F} \approx \mathcal{O}_{\mathbb{P}^4}(1)$ for any fiber F; or*
5. *$n = 5$, X is a singular 2-Gorenstein Fano 5-fold described in* [7, (7.7.5)]*; or*
6. *X is a Fano n-fold with $K_X \approx -(n-3)\mathcal{K}$ if n is even and $2K_X \approx -2(n-3)\mathcal{K}$ if n is odd.*

We now consider the case $n = 4$. From [4, (2.1)] we know that $K_X + 3\mathcal{K}$ is nef and big unless either

i) $(X, \mathcal{K}) \cong (\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(1))$; or
ii) X is a Gorenstein Fano 4-fold with $-K_X \approx 3\mathcal{K}$ and $-4\mathcal{K} \approx 6\mathcal{D}$; or
iii) there exists a holomorphic map $\psi : X \to C$, where C is a smooth curve, $4K_X + 6\mathcal{D} \approx \psi^* H$ for some ample line bundle H on C. In this case $K_X + 3\mathcal{K} \approx \psi^* H$, i.e., $(X, \mathcal{K})$ is a quadric fibration over C; or
iv) there exists a holomorphic map $\psi : X \to S$, where S is a smooth surface, $4K_X + 6\mathcal{D} \approx \psi^* H$ for some ample line bundle H on S. In this case $K_X + 3\mathcal{K} \approx \psi^* H$, i.e., $(X, \mathcal{K})$ is a scroll over S.

In case i), $d_4 = 2$. In cases iii) and iv), $4K_X + 6\mathcal{D}$ is effective. From [7, (7.6.1)] we know that $H^0(X, 4K_X + 6\mathcal{D}) \cong H^0(M, 4K_M + 6L)$, so that $4K_M + 6L = 4(K_M + 2L) - 2L$ is effective on M. Therefore, dotting with $(K_M + 2L)^j \cdot L^{3-j}$, $j = 0, 1, 2, 3$, we obtain $2d_j > d_{j-1}$, $j = 1, 2, 3, 4$.

The arguments above show the following result.

Theorem 2.2. *Let $\mathcal{M}$ be a smooth connected 4-fold polarized by a very ample line bundle $\mathcal{L}$. Assume that $\kappa(K_{\mathcal{M}} + 2\mathcal{L}) = 4$. Let (M, L), $(X, \mathcal{D})$, $\mathcal{K} \approx K_X + 2\mathcal{D}$ be the first and the second reduction of $(\mathcal{M}, \mathcal{L})$ as in (1.1). Let d_j, $j = 0, 1, 2, 3, 4$, $d_0 = d$, be the pluridegrees of (M, L). Assume that $K_X + 3\mathcal{K}$ is not nef and big. Then either*

1. *$2d_j > d_{j-1}$, $j = 1, 2, 3, 4$; or*
2. *$(X, \mathcal{K}) \cong (\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(1))$, $d_4 = 1$; or*
3. *X is a Gorenstein Fano 4-fold with $-K_X \approx 3\mathcal{K}$ and $-4K_X \approx 6\mathcal{D}$.*

3. The case $n \geq 6$

Let $\mathcal{M}$ be a smooth connected n-fold polarized by a very ample line bundle $\mathcal{L}$. Assume that $\kappa(K_{\mathcal{M}} + (n-2)\mathcal{L}) = n$. Let $\pi : (\mathcal{M}, \mathcal{L}) \to (M, L)$, $\varphi : (M, L) \to (X, \mathcal{D})$, $\mathcal{K} \approx K_X + (n-2)\mathcal{D}$, be the first and the second reduction of $(\mathcal{M}, \mathcal{L})$ as in (1.1). In this section we consider the general case when $n \geq 6$.

From the results of §2 we can assume that $K_X + (n-3)\mathcal{K}$ is nef and big. By Proposition (1.2) this nefness and bigness condition together with the assumption $\kappa(K_{\mathcal{M}} + (n-2)\mathcal{L}) = n$ is equivalent to the condition $\kappa(K_{\mathcal{M}} + (n-3)\mathcal{L}) = n$.

It should be noted that $K_X + (n-3)\mathcal{K}$ is always nef under the assumption $\kappa(K_{\mathcal{M}} + (n-2)\mathcal{L}) = n$ if $n \geq 7$ (see [1, (3.1)]). We can prove the following general result.

Theorem 3.1. *Let $\mathcal{M}$ be a smooth connected n-fold polarized by a very ample line bundle $\mathcal{L}$, with $n \geq 6$. Assume that $\kappa(K_{\mathcal{M}} + (n-3)\mathcal{L}) = n$. Let $\pi : (\mathcal{M}, \mathcal{L}) \to (M, L)$, $\varphi : (M, L) \to (X, \mathcal{D})$, be the first and the second reduction of $(\mathcal{M}, \mathcal{L})$. Let d_j be the pluridegrees of (M, L), $j = 0, \ldots, n$, $d_0 = d$. Then either*

1. $d_j \geq 2\left(\frac{n-1}{n+1}\right) d_{j-1}$, $j = 1, \ldots, n$, *and hence* $d_n \geq \left(\frac{2n-2}{n+1}\right)^n d$; *or*
2. $n = 8$ *and there is a birational morphism* $\psi : X \to \mathbb{P}^8$ *such that* $K_X + (n-3)\mathcal{D} \approx \psi^*\mathcal{O}_{\mathbb{P}^8}(1)$; *or*
3. $n = 6$, $\widehat{d} = \mathcal{L}^6 = 364$ *and* $d_j \geq \frac{13}{16} d_{j-1}$, $j = 1, \ldots, 6$.

Proof. First assume $n \geq 7$. Thus Theorem (3.1) of [1] applies to say that $(n+1)K_{\mathcal{M}} + n(n-3)\mathcal{L}$ is effective unless $n = 8$ as in case 2). From [7, (7.6.1)] we know that

$$H^0(\mathcal{M}, (n+1)K_{\mathcal{M}} + n(n-3)\mathcal{L}) \cong H^0(M, (n+1)K_M + n(n-3)L).$$

Therefore, up to the special case with $n = 8$,

$$(n+1)K_M + n(n-3)L = (n+1)(K_M + (n-2)L) - 2(n-1)L$$

is effective. Thus, dotting with $(K_M + (n-2)L)^{j-1} \cdot L^{n-j}$, we get

$$d_j \geq 2\left(\frac{n-1}{n+1}\right) d_{j-1}, \quad j = 1, \ldots, n.$$

Hence in particular

$$d_n \geq \left(\frac{2n-2}{n+1}\right)^n d.$$

Assume now $n = 6$. Thus from [1, (4.1)] we know that $13K_{\mathcal{M}} + 36\mathcal{L}$ is effective. Moreover, if $\widehat{d} = \mathcal{L}^6 \neq 364$, then $7K_{\mathcal{M}} + 18\mathcal{L}$ is effective.

Let us consider first the general case $\widehat{d} \neq 364$. From [7, (7.6.1)] we know that

$$H^0(\mathcal{M}, 7K_{\mathcal{M}} + 18\mathcal{L}) \cong H^0(M, 7K_M + 18L).$$

Therefore $7K_M + 18L = 7(K_M + 4L) - 10L$ is effective. Thus dotting with $7(K_M + (n-2)L)^{j-1} \cdot L^{6-j}$, we get

$$d_j \geq \frac{10}{7} d_{j-1} = \left(\frac{2n-2}{n+1}\right) d_{j-1}, \quad j = 1, \ldots, 6, \tag{3}$$

as in 1). In the special case $\widehat{d} = 364$, the same argument gives

$$13 d_j \geq 16 d_{j-1}, \quad j = 1, \ldots, 6, \tag{4}$$

as in case 3). Q.E.D.

Remark 3.2. Notation and assumptions as in (3.1). Assume that either X is not Gorenstein or $\mathcal{D}$ is not Cartier. Then we know from [1, (3.1)] that $K_{\mathcal{M}} + (n-4)\mathcal{L}$ is effective. By using again [7, (7.6.1)] we thus conclude that $K_M + (n-4)L = K_M + (n-2)L - 2L$ is effective. Therefore, dotting with $(K_M + (n-2)L)^{j-1} \cdot L^{n-j}$, we get the stronger bound $d_j > 2d_{j-1}$, $j = 1, \ldots, n$, and hence $d_n > 2^n d$.

We have the following numerical conditions, expressed in terms of pluridegrees, for a given variety $\mathcal{V}$ to be a k-fold section of the polarized pair $(\mathcal{M}, \mathcal{L})$.

Remark 3.3. Let $\mathcal{M}$ be a smooth connected n-fold polarized by a very ample line bundle $\mathcal{L}$ with either $n \geq 7$ or $n = 6$ and $\mathcal{L}^6 \neq 364$, as in (3.1). Let $\mathcal{V}$ be the k-fold section of $\mathcal{M}$ obtained as the transversal intersection of $n-k$ general members of $|\mathcal{L}|$, $k = 2, \ldots, n-1$. Let $\mathcal{L}_{\mathcal{V}}$ be the restriction of $\mathcal{L}$ to $\mathcal{V}$. Let $\pi : (\mathcal{M}, \mathcal{L}) \to (M, L)$, $\varphi : (M, L) \to (X, \mathcal{D})$ be the first and the second reduction of $(\mathcal{M}, \mathcal{L})$. Let $V := \pi(\mathcal{V})$, $Y := \varphi(V)$. Let L_V and $\mathcal{D}_Y$ be the restrictions of L and

$\mathcal{D}$ to V and Y respectively. Thus intersections are compatible with reductions, i.e., (V, L_V), $(Y, \mathcal{D}_Y)$ are the first and the second reduction of $(\mathcal{V}, \mathcal{L}_{\mathcal{V}})$.

Let the d_j's be the pluridegrees of (M, L) and let

$$d_j(V) := (K_V + (k-2)L_V)^j \cdot L_V^{k-j} (= (K_M + (n-2)L^j) \cdot L^{n-j})$$

be the pluridegrees of (V, L_V). We have $d_k = d_k(V)$, $k \geq 1$, and $d = L^n = L_V^k$.

Assuming $n \neq 8$ and $\mathcal{L}^n \neq 364$, we have from (3.1)

$$d_k(V) = d_k \geq \left(\frac{2n-2}{n+1}\right)^k d, \tag{5}$$

i.e., $\mathcal{V}$ cannot be a k-fold section of $(\mathcal{M}, \mathcal{L})$ as soon as $d_k(V) = d_k < \left(\frac{2n-2}{n+1}\right)^k d$.

In particular, for $k = 2$, inequality (5) gives a lower bound for the Euler characteristic of the surface section $V = S$. Indeed Miyaoka's inequality $d_2 < 9\chi(\mathcal{O}_S)$ combined with (5) gives

$$\chi(\mathcal{O}_S) > \frac{4}{9}\left(\frac{n-1}{n+1}\right)^2 d. \tag{6}$$

4. The case $n = 5$

Let $\mathcal{M}$ be a smooth connected 5-fold polarized by a very ample line bundle $\mathcal{L}$. Assume that $\kappa(K_{\mathcal{M}} + 3\mathcal{L}) = 5$. Let $\pi : (\mathcal{M}, \mathcal{L}) \to (M, L)$, $\varphi : (M, L) \to (X, \mathcal{D})$, $\mathcal{K} \approx K_X + 3\mathcal{D}$, be the first and the second reduction of $(\mathcal{M}, \mathcal{L})$ as in (1.1).

From the results of §2 we can assume that $K_X + 2\mathcal{K}$ is nef and big. By Proposition (1.2) this nefness and bigness condition together with the assumption $\kappa(K_{\mathcal{M}} + 3\mathcal{L}) = 5$ is equivalent to the condition $\kappa(K_{\mathcal{M}} + 2\mathcal{L}) = 5$.

We can prove the following result. In many proofs special arguments are needed for the cases when projective invariants are small. The inequalities in Theorems (4.1) and (5.1) are very useful in such situations.

Theorem 4.1. *Let $\mathcal{M}$ be a smooth connected 5-fold polarized by a very ample line bundle $\mathcal{L}$. Assume that $\kappa(K_{\mathcal{M}} + 2\mathcal{L}) = 5$. Let $\pi : (\mathcal{M}, \mathcal{L}) \to (M, L)$, $\varphi : (M, L) \to (X, \mathcal{D})$, $\mathcal{K} \approx K_X + 3\mathcal{D}$, be the first and the second reduction of $(\mathcal{M}, \mathcal{L})$. Let d_j's, $d_0 = d$, $j = 0, \ldots, 5$, be the pluridegrees of (M, L). Then either*

1. *the inequalities $d \geq 14$, $d_1 \geq 19$, $d_2 \geq 24$, $d_3 \geq 30$, $d_4 \geq 37$, $d_5 \geq 44$ are all satisfied; or*
2. *$\mathcal{M}$ is the complete intersection of a quadric and a quintic in $\mathbb{P}^7$, $\mathcal{L} \cong \mathcal{O}_{\mathcal{M}}(1)$, $(\mathcal{M}, \mathcal{L}) \cong (M, L)$, $d = 10$, $d_j = 2^j d$, $j = 1, 2, 3, 4, 5$; or*
3. *$\mathcal{M}$ is the complete intersection of a quadric and a sextic in $\mathbb{P}^7$, $\mathcal{L} \cong \mathcal{O}_{\mathcal{M}}(1)$, $(\mathcal{M}, \mathcal{L}) \cong (M, L)$, $d = 12$, $d_j = 3^j d$, $j = 1, 2, 3, 4, 5$; or*
4. *$\mathcal{M}$ is the complete intersection of a cubic and a quartic in $\mathbb{P}^7$, $\mathcal{L} \cong \mathcal{O}_{\mathcal{M}}(1)$, $(\mathcal{M}, \mathcal{L}) \cong (M, L)$, $d = 12$, $d_j = 2^j d$, $j = 1, 2, 3, 4, 5$.*

Proof. We can assume that $h^0(\mathcal{L}) \geq 8$ since otherwise the result is trivial. Let $\varphi : (M, L) \to (X, \mathcal{D})$, $\mathcal{K} \approx K_X + 3\mathcal{D}$, be the second reduction of $(\mathcal{M}, \mathcal{L})$. Consider the line bundle $\mathcal{F} := 2(K_X + 2\mathcal{D})$ on X. Since $K_X + 2\mathcal{K} \approx 3(K_X + 2\mathcal{D})$ is nef and big (see (1.2)), we conclude that

$$\mathcal{F} \text{ is nef and big.} \tag{7}$$

Moreover $2\mathcal{F} - K_X = 3K_X + 8\mathcal{D} = K_X + 2\mathcal{D} + 2(K_X + 3\mathcal{D})$ is nef and big since, by the above, $K_X + 2\mathcal{D}$ is nef and big and $\mathcal{K} \approx K_X + 3\mathcal{D}$ is ample. Thus the

Kawamata-Shokurov basepoint free theorem (see e.g., [7, (1.5.1)]) applies to say that

(8) $t\mathcal{F}$ is spanned by global sections for sufficiently large integers t.

From [7, (7.2.9)] (see also [1, (0.2.8)]) we know that $h^i(t(K_X+2\mathcal{D}))=0$ for all positive integers t and for $i>0$. Then in particular

$$h^i(t\mathcal{F})=0 \text{ for } i>0,\ t>0. \tag{9}$$

Thus in view of conditions (7), (8), (9), Proposition (0.5) of [1] applies to say that either $h^0(K_X+5\mathcal{F})=h^0(11K_X+20\mathcal{D})>0$, or there exists a birational morphism $\psi : X \to \mathbb{P}^5$ such that $\mathcal{F}=2(K_X+2\mathcal{D})\approx \psi^*\mathcal{O}_{\mathbb{P}^5}(1)$. The latter case is easily ruled out. Indeed we have $(\psi_* K_X)^{**}\approx \mathcal{O}_{\mathbb{P}^5}(-6)$ and $(\psi_*\mathcal{D})^{**}\approx \mathcal{O}_{\mathbb{P}^5}(a)$ for some integer a. Hence we get the numerical contradiction $-12+4a=1$.

Thus we conclude that $11K_X+20\mathcal{D}$ is effective. We claim that $2(11K_M+20L)$ is effective. To see this recall that $2K_M\approx\varphi^*(2K_X)+3\Delta$ and $2L\approx\varphi^*(2\mathcal{D})-\Delta$, for some effective Cartier divisor Δ (see (1.1)). Thus

$$22K_M+40L\approx\varphi^*(22K_X+40\mathcal{D})+(33-20)\Delta.$$

Hence $2(11K_M+20L)\approx 22\mathcal{K}_M-26L$ is effective, where $\mathcal{K}_M := K_M+3L$. Thus dotting with $\mathcal{K}_M^{j-1}\cdot L^{5-j}$, we get

$$11d_j\geq 13d_{j-1},\quad j=1,2,3,4,5. \tag{10}$$

From Lemma (1.5) we know that $d\geq\widehat{d}\geq 8$. Then inequalities (10) give $d_1\geq 10$.

Assume that $|\mathcal{L}|$ embeds $\mathcal{M}$ in $\mathbb{P}^N$, $N\geq 9$. Suppose $\widehat{d}:=\mathcal{L}^5=14$. Then $d\geq 14$ and $d_1\geq 17$ by (10). Castelnuovo's bound [8, (1.9)] gives $g:=g(\mathcal{L})\leq$ Castel$(14,5)\leq 15$. Then we find the numerical contradiction $31\leq d_1+d=\widehat{d}+\widehat{d}_1=2g-2\leq 28$.

Exactly the same argument, by using (10) and Castelnuovo's bound, rules out the low degree cases $8\leq\widehat{d}\leq 13$.

Thus $d\geq\widehat{d}\geq 15$. From (10) we obtain

$$d\geq 15,\ d_1\geq 18,\ d_2\geq 22,\ d_3\geq 26,\ d_4\geq 31,\ d_5\geq 37.$$

If $d_1=18$, then $d\geq 16$ by parity, contradicting $d_1^2=324\geq dd_2\geq 330$. Therefore $d_1\geq 19$. Then $d_2^2\geq d_1d_3\geq 494$ yields $d_2\geq 23$. If $d_3=26$, then $d_2\geq 24$ by parity, contradicting $d_3^2=676\geq d_2d_4\geq 744$. Therefore $d_3\geq 27$. If $d_3=27$, then $d_2\geq 25$ by parity, which contradicts $d_3^2=729\geq d_2d_4\geq 775$. Thus $d_3\geq 28$, so that (10) gives $d_4\geq 34$ and $d_5\geq 41$. Summarizing, we have

$$d\geq 15,\ d_1\geq 19,\ d_2\geq 23,\ d_3\geq 28,\ d_4\geq 34,\ d_5\geq 41.$$

From $d_2^2\geq d_1d_3\geq 532$ we get $d_2\geq 24$, so that (10) gives $d_3\geq 29$, $d_4\geq 35$, $d_5\geq 42$. If $d_3=29$, then $d_2\geq 25$ by parity, contradicting $d_3^2=841\geq d_2d_4\geq 875$. Thus $d_3\geq 30$, so that (10) yields $d_4\geq 36$, $d_5\geq 43$. Then we have

$$d\geq 15,\ d_1\geq 19,\ d_2\geq 24,\ d_3\geq 30,\ d_4\geq 36,\ d_5\geq 43.$$

Finally, if $d_4=36$, then $d_5\geq 44$ by parity, which contradicts $d_4^2=1296\geq d_3d_5\geq 1320$. Therefore $d_4\geq 37$ and (10) gives $d_5\geq 44$, as in case 1).

Thus we can assume that $|\mathcal{L}|$ embeds $\mathcal{M}$ in $\mathbb{P}^N$ with $7\leq N\leq 8$. Then by the Barth-Lefschetz theorem we know that $\mathrm{Pic}(\mathcal{M})\cong\mathbb{Z}[\mathcal{L}]$. Note that this condition implies $\mathcal{M} \cong M \cong X$. Moreover $\mathcal{L} \approx \mathcal{O}_{\mathcal{M}}(1)$, $K_{\mathcal{M}} \approx \mathcal{O}_{\mathcal{M}}(r)$ for some $r \in \mathbb{Z}$ and hence

$$d_j = (r+3)d_{j-1} = (r+3)^j d, \quad j = 1,2,3,4,5. \tag{11}$$

From (10) one has $r+3 \geq 2$. Therefore we have $2g-2 = d+d_1 \geq 3d$ and hence using Castelnuovo's inequality we have $d \geq 10$. Hence $d_1 \geq 12$ by (10). Let $\mathcal{V}$ be the smooth 3-fold section of $\mathcal{M}$ obtained as the transversal intersection of two general members of $|\mathcal{L}|$. Let $\mathcal{L}_{\mathcal{V}}$ be the restriction of $\mathcal{L}$ to $\mathcal{V}$. Then $\mathcal{V}$ is embedded by $|\mathcal{L}_{\mathcal{V}}|$ in $\mathbb{P}^{N-2}$ as 3-fold of degree $d = \widehat{d} = \mathcal{L}_{\mathcal{V}}^3$. Moreover $\kappa(K_{\mathcal{V}} + \mathcal{L}_{\mathcal{V}}) = 3$ since $\kappa(K_{\mathcal{M}} + 3\mathcal{L}) = 5$ and $(\mathcal{V}, \mathcal{L}_{\mathcal{V}})$ coincides with its own first reduction since $(\mathcal{M}, \mathcal{L}) \cong (M, L)$. In what follows we use the lists from [3, Chapter 6] of 3-folds of degree ≤ 12 in $\mathbb{P}^5$.

Let $d = 10$. Then $\mathcal{M}$ lies in $\mathbb{P}^7$ since otherwise Castelnuovo's bound [8, (1.9)] gives $g \leq \mathrm{Castel}(10,4) \leq 9$, which leads to the contradiction $22 \leq d + d_1 = 2g-2 \leq 16$. Since by (11) all the pluridegrees are multiples of $d = 10$, we see that the only possible case is when $\mathcal{V}$ is the complete intersection of a quadric and a quintic in $\mathbb{P}^5$. Then $\mathcal{M}$ is the complete intersection of a quadric and a quintic in $\mathbb{P}^7$, and $d_j = 2^j d$, $j = 1,2,3,4,5$ as in case 2).

Thus we can assume $d \geq 11$. Note that the case $d = 11$ doesn't happen. Indeed in this case d_1 must be odd by parity and hence by (11) we have $d_1 = (2k+1)d$ for some positive integer k. Then in particular $d_1 \geq 3d$. Since d is odd we have Castelnuovo's inequality $2g-2 \leq \frac{(d-1)(d-3)}{2} - 2$. This gives the numerical contradiction $38 \geq 2g-2 = d_1 + d \geq 4d \geq 44$.

Thus $d \geq 12$. Assume that $d = 12$ and that $|\mathcal{L}|$ embeds $\mathcal{M}$ in $\mathbb{P}^7$. Then $|\mathcal{L}_{\mathcal{V}}|$ embeds $\mathcal{V}$ in $\mathbb{P}^5$ and, recalling (11), we see that either $\mathcal{V}$ is the complete intersection of a quadric and a sextic in $\mathbb{P}^5$ or $\mathcal{V}$ is the complete intersection of a cubic and a quartic in $\mathbb{P}^5$. Accordingly, either $\mathcal{M}$ is the complete intersection of a quadric and a sextic in $\mathbb{P}^7$, with $\mathcal{L} \cong \mathcal{O}_{\mathcal{M}}(1)$, $d_j = 3^j d$, $j = 1,2,3,4,5$, as in case 3), or $\mathcal{M}$ is the complete intersection of a cubic and a quartic in $\mathbb{P}^7$, with $\mathcal{L} \cong \mathcal{O}_{\mathcal{M}}(1)$, $d_j = 2^j d$, $j = 1,2,3,4,5$, as in case 4).

Thus we can assume that $d = 12$ and that $|\mathcal{L}|$ embeds $\mathcal{M}$ in $\mathbb{P}^8$. Then Castelnuovo's bound gives $g \leq \mathrm{Castel}(12,4) \leq 15$. Since, by (11), $d_1 = (r+3)d$ and $r+3 \geq 2$ we find the contradiction $36 \leq 3d \leq d + d_1 = 2g-2 \leq 30$.

Thus $d \geq 13$. Assume that $d = 13$ and that $|\mathcal{L}|$ embeds $\mathcal{M}$ in $\mathbb{P}^7$. Since d_1 must be odd by parity, we have $d_1 = (2k+1)d$ for some positive integer k from (11). Castelnuovo's bound gives $g \leq \mathrm{Castel}(13,3) \leq 30$. If $k \geq 2$, i.e., $d_1 \geq 5d$, we find the contradiction $78 \leq 6d = d + d_1 = 2g-2 \leq 58$. Therefore $k = 1$, i.e., $d_1 = 3d$, so that $g = 2d+1 = 27$ doesn't reach the maximum according to Castelnuovo's bound. Thus, since the hyperplane curve section of $\mathcal{M}$ lies in $\mathbb{P}^3$, we can use Gruson-Peskine bound which gives (see e.g., [7, (1.4.9)])

$$g \leq \frac{d(d-3)}{6} + 1 = \frac{68}{3},$$

or $g \leq 22$. This contradicts the above equality $g = 27$.

Therefore we have to consider the case when $d = 13$ and $|\mathcal{L}|$ embeds $\mathcal{M}$ in $\mathbb{P}^8$. Again, as above, $d_1 \geq 3d$. Castelnuovo's bound yields $g \leq \mathrm{Castel}(13,4) \leq 18$ and then we get the contradiction $52 \leq 4d \leq d_1 + d = 2g-2 \leq 34$.

Thus we can assume $d \geq 14$, so that (11) gives, since $r+3 \geq 2$, the inequalities $d \geq 14$, $d_1 \geq 28$, $d_2 \geq 56$, $d_3 \geq 112$, $d_4 \geq 224$, $d_5 \geq 448$ and we fall again in case 1). Q.E.D.

5. The case $n = 4$

Let $\mathcal{M}$ be a smooth connected 4-fold polarized by a very ample line bundle $\mathcal{L}$. Assume that $\kappa(K_{\mathcal{M}} + 2\mathcal{L}) = 4$. Let $\pi : (\mathcal{M}, \mathcal{L}) \to (M, L)$, $\varphi : (M, L) \to (X, \mathcal{D})$, $\mathcal{K} \approx K_X + 2\mathcal{D}$, be the first and the second reduction of $(\mathcal{M}, \mathcal{L})$ as in (1.1).

From the results of §2 we can assume that $K_X + 3\mathcal{K}$ is nef and big. By Proposition (1.3) this is equivalent to $\kappa(K_X + 3\mathcal{K}) = 4$, or, since $K_X + 3\mathcal{K} \approx 4(K_X + \frac{3}{2}\mathcal{D})$, to $\kappa(K_X + \frac{3}{2}\mathcal{D}) = 4$. By noting that

$$K_{\mathcal{M}} + \frac{3}{2}\mathcal{L} \approx \pi^*\varphi^*(K_X + \frac{3}{2}\mathcal{D}) + Z,$$

for some effective $\mathbb{Q}$-Cartier divisor Z on $\mathcal{M}$ (see (1.1)), we conclude that the nefness and bigness condition above is equivalent to $\kappa(K_{\mathcal{M}} + \frac{3}{2}\mathcal{L}) = 4$.

Theorem 5.1. *Let $\mathcal{M}$ be a smooth connected 4-fold polarized by a very ample line bundle $\mathcal{L}$. Assume that $\kappa(K_{\mathcal{M}} + \frac{3}{2}\mathcal{L}) = 4$. Let $\pi : (\mathcal{M}, \mathcal{L}) \to (M, L)$ be the first reduction of $(\mathcal{M}, \mathcal{L})$. Let d_j's, $d_0 = d$, $j = 0, 1, 2, 3, 4$, be the pluridegrees of (M, L). Then either*

1. *the inequalities $d \geq 11$, $d_1 \geq 9$, $d_2 \geq 7$, $d_3 \geq 5$, $d_4 \geq 3$ are all satisfied; or*
2. *$\mathcal{M}$ is the complete intersection of a quadric and a quartic in $\mathbb{P}^6$, $\mathcal{L} \cong \mathcal{O}_{\mathcal{M}}(1)$, $(\mathcal{M}, \mathcal{L}) \cong (M, L)$, and $d_j = 8$, $j = 0, 1, 2, 3, 4$; or*
3. *$\mathcal{M}$ is the complete intersection of two cubics in $\mathbb{P}^6$, $\mathcal{L} \cong \mathcal{O}_{\mathcal{M}}(1)$, $(\mathcal{M}, \mathcal{L}) \cong (M, L)$, and $d_j = 9$, $j = 0, 1, 2, 3, 4$; or*
4. *$\mathcal{M}$ is the complete intersection of a quadric and a quintic in $\mathbb{P}^6$, $\mathcal{L} \cong \mathcal{O}_{\mathcal{M}}(1)$, $(\mathcal{M}, \mathcal{L}) \cong (M, L)$, and $d = 10$, $d_j = 2d_{j-1}$, $j = 1, 2, 3, 4$.*

Proof. Let $\varphi : (M, L) \to (X, \mathcal{D})$, $\mathcal{K} \approx K_X + 2\mathcal{D}$, be the second reduction of $(\mathcal{M}, \mathcal{L})$. Since $K_X + 3\mathcal{K} \approx 4\mathcal{K} - 2\mathcal{D}$ is nef and big, dotting with $\mathcal{K}^j \cdot \mathcal{D}^{3-j}$, $j = 2, 3$, we obtain $d_4' > d_3'/2$, $d_3' > d_2'/2$ on X. Therefore, by using (1), we have $d_4 > d_3/2$, $d_3 > d_2/2$ on M. Thus Hodge index inequalities [8, Eq. (2)] give $d_2 > d_1/2$, $d_1 > d/2$. Then we get

$$d_4 > \frac{d_3}{2} > \frac{d_2}{4} > \frac{d_1}{8} > \frac{d}{16}. \tag{12}$$

In particular $d_4 \geq 2$ since otherwise $d = \widehat{d} = 1$ and hence $(\mathcal{M}, \mathcal{L}) \cong (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$, contradicting the condition $\kappa(K_{\mathcal{M}} + 2\mathcal{L}) = 4$. From Lemma (1.5) we know that $d \geq \widehat{d} := \mathcal{L}^4 \geq 8$. Then the inequalities (12) give $d \geq 8$, $d_1 \geq 5$, $d_2 \geq 3$, $d_3 \geq 2$, $d_4 \geq 2$. By using repeatedly the Hodge index inequalities, we obtain $d_1 \geq 7$, $d_2 \geq 6$, $d_3 \geq 4$. Then $d_4 > d_3/2$ from (12) gives $d_4 \geq 3$, and hence $d_3^2 \geq d_2 d_4 = 18$ yields $d_3 \geq 5$. If $d_1 = 7$, then $d \geq 9$ by parity. This contradicts $d \leq d_1^2/d_2 \leq 49/6 < 9$. Therefore $d_1 \geq 8$, so that $d_2^2 \geq d_1 d_3 \geq 40$ gives $d_2 \geq 7$. Thus we have

$$d \geq 8, \quad d_1 \geq 8, \quad d_2 \geq 7, \quad d_3 \geq 5, \quad d_4 \geq 3. \tag{13}$$

Assume now that $|\mathcal{L}|$ embeds $\mathcal{M}$ in $\mathbb{P}^N$, $N \geq 7$. If $\widehat{d} = \mathcal{L}^4 = 8, 9, 10$, Castelnuovo's bound $g := g(\mathcal{L}) \leq \mathrm{Castel}(\widehat{d}, 4)$ gives respectively $g \leq 5$, $g \leq 7$, $g \leq 9$. In each case $18 \leq d_1 + d = \widehat{d}_1 + \widehat{d} = 2g - 2 \leq 16$ leads to a numerical contradiction. Thus $d \geq \widehat{d} \geq 11$ and $d_1^2 \geq dd_2 \geq 77$ gives $d_1 \geq 9$. Then, recalling (13), we are in case 1).

Thus we can assume that $|\mathcal{L}|$ embeds $\mathcal{M}$ in $\mathbb{P}^6$. Therefore by the Barth-Lefschetz theorem we know that $\mathrm{Pic}(\mathcal{M}) \cong \mathbb{Z}[\mathcal{L}]$. Note that this condition implies $\mathcal{M} \cong M \cong X$. Moreover $\mathcal{L} \cong \mathcal{O}_{\mathcal{M}}(1)$, $K_{\mathcal{M}} \cong \mathcal{O}_{\mathcal{M}}(r)$ for some $r \in \mathbb{Z}$ and hence

$$d_j = (r+2)d_{j-1} = (r+2)^j d, \quad j = 1,2,3,4. \tag{14}$$

Let $\mathcal{V}$ be a general smooth member of $|\mathcal{L}|$. Let $\mathcal{L}_{\mathcal{V}}$ be the restriction of $\mathcal{L}$ to $\mathcal{V}$. Then $\mathcal{V}$ is a smooth 3-fold of degree $d = \widehat{d} = \mathcal{L}_{\mathcal{V}}^3$ embedded by $|\mathcal{L}_{\mathcal{V}}|$ in $\mathbb{P}^5$. Moreover $\kappa(K_{\mathcal{V}} + \mathcal{L}_{\mathcal{V}}) = 3$ since $\kappa(K_{\mathcal{M}} + 2\mathcal{L}) = 4$ and $(\mathcal{V}, \mathcal{L}_{\mathcal{V}})$ coincides with its own first reduction since $(\mathcal{M}, \mathcal{L}) \cong (M, L)$. In what follows we use the lists of 3-folds of degree ≤ 12 in $\mathbb{P}^5$ from [3, Chapter 6].

Assume $d(=\widehat{d}) = 8$. There are only two possible cases. In the first case $K_{\mathcal{V}} + \mathcal{L}_{\mathcal{V}}$ is not nef and big, contradicting the assumption $\kappa(K_{\mathcal{V}} + \mathcal{L}_{\mathcal{V}}) = 3$. In the second case $\mathcal{V}$ is the complete intersection of a quadric and a quartic in $\mathbb{P}^5$, so that $\mathcal{M}$ is a complete intersection of type $(2,4)$ in $\mathbb{P}^6$ and $d_j = 8$, $j = 0,1,2,3,4$, as in 2).

Let $d(=\widehat{d}) = 9$. Then from the lists we see that $\widehat{d}_1 = d_1 \leq 13$ and hence (14) gives $r + 2 = 1$, i.e., $d_j = d_{j-1}$, $j = 1,2,3,4$. This is not possible unless $\mathcal{V}$ is the complete intersection of two cubic hypersurfaces in $\mathbb{P}^5$. In this case $\mathcal{M}$ is the complete intersection of two cubics in $\mathbb{P}^6$ and $d_j = 9$, $j = 0,1,2,3,4$, as in 3).

Assume $d(=\widehat{d}) = 10$. Then either $\mathcal{V}$ is the complete intersection of a quadric and a quintic in $\mathbb{P}^5$, so that $\mathcal{M}$ is a complete intersection of type $(2,5)$ in $\mathbb{P}^6$, $K_{\mathcal{M}} \approx \mathcal{O}_{\mathcal{M}}$, $r = 0$, $d_j = 2d_{j-1}$, $j = 1,2,3,4$, as in case 4), or $d_3 = 1$, which contradicts (13), or $d_1 = d_2 = d_3 = d_4 = 12$. From (14) we have $d_j = (r+2)^j d = (r+2)^j 10$, $j = 1,2,3,4$, so we find again a contradiction.

Thus we conclude that $d(=\widehat{d}) \geq 11$ and $d_1^2 \geq dd_2 \geq 77$ gives $d_1 \geq 9$. Then, recalling (13), we fall again in case 1). Q.E.D.

6. The case $n = 3$, I

Let $\mathcal{M}$ be a smooth connected 3-fold polarized by a very ample line bundle $\mathcal{L}$. Assume that $\kappa(K_{\mathcal{M}} + \mathcal{L}) = 3$, i.e., $(\mathcal{M}, \mathcal{L})$ is of log-general type. Let $\pi : (\mathcal{M}, \mathcal{L}) \to (M, L)$, $\varphi : (M, L) \to (X, \mathcal{D})$, $\mathcal{K} \approx K_X + \mathcal{D}$, be the first and the second reduction of $(\mathcal{M}, \mathcal{L})$ as in (1.1). Let d_j, $j = 0,1,2,3$, $d_0 = d$, be the pluridegrees of (M, L).

First, let us state some general relations. Let S be a smooth general member of $|L|$. Set $\mathcal{K}_M := K_M + L$ and $h := h^0(K_M + L)$. Recall that $h \geq 2$ from [6, (1.2)]. Since $\mathcal{K}_M$ is nef and big, the Riemann-Roch formula yields

$$\begin{aligned} h^0(K_M + t\mathcal{K}_M) &= -\chi(t\mathcal{K}_M) \\ &= \frac{t^3 d_3}{6} + \frac{t^2(d_3 - d_2)}{4} \\ &\quad + \frac{t\mathcal{K}_M \cdot ((\mathcal{K}_M - L)^2 + c_2(M))}{12} - \chi(\mathcal{O}_M). \end{aligned}$$

Since $\mathcal{K}_M \cdot (\mathcal{K}_M - L)^2 = d_3 - 2d_2 + d_1$,

$$\mathcal{K}_M \cdot c_2(M) = K_M \cdot c_2(M) + L \cdot c_2(M) = -24\chi(\mathcal{O}_M) + 12\chi(\mathcal{O}_S) - d_2 - d_1,$$

and $\chi(\mathcal{O}_M) = \chi(\mathcal{O}_S) - h$, we find

$$h^0(K_M + t\mathcal{K}_M) = \frac{t^3 d_3}{6} + \frac{t^2 d_3}{4} - \frac{t^2 d_2}{4} + \frac{t d_3}{12} - \frac{t d_2}{4} - (t+1)\chi(\mathcal{O}_S) + h(2t+1).$$

For $t = 1$ we have

$$2h^0(K_M + \mathcal{K}_M) + 4\chi(\mathcal{O}_S) + d_2 = d_3 + 6h. \tag{15}$$

For $t = 2$ we have

$$2h^0(K_M + 2\mathcal{K}_M) + 6\chi(\mathcal{O}_S) + 3d_2 = 5d_3 + 10h. \tag{16}$$

For $t = 3$ we have

$$h^0(K_M + 3\mathcal{K}_M) + 4\chi(\mathcal{O}_S) + 3d_2 = 7d_3 + 7h. \tag{17}$$

We have the following four exceptions to $K_X + 2\mathcal{K}$ being nef from [1, (2.2)].

i) $d_3 = 1$, $(X, \mathcal{D}) \cong (\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(5))$.
ii) $d_3 = 2$, $(X, \mathcal{D}) \cong (\mathcal{Q}, \mathcal{O}_{\mathcal{Q}}(1))$, where $\mathcal{Q}$ is a quadric in $\mathbb{P}^4$ and $\mathcal{O}_{\mathcal{Q}}(1)$ is the restriction of $\mathcal{O}_{\mathbb{P}^4}(1)$ to $\mathcal{Q}$, under the usual embedding $\mathcal{Q} \subset \mathbb{P}^4$.
iii) $d_3 = 4$, and X is the cone over $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$ with $2\mathcal{D} \approx 7\xi$, where ξ is the tautological line bundle of the cone.
iv) X is a $\mathbb{P}^2$-bundle, $p : X \to C$, over a smooth curve C with $K_X + 3\mathcal{K} \approx p^*H$ for some ample line bundle H on C.

Now let us analyze the special case iv) of a variety $\mathcal{M}$ with second reduction a $\mathbb{P}^2$-bundle X over a curve C of genus $q = h^1(\mathcal{O}_X)$ and $\mathcal{K}_{|F} \approx \mathcal{O}_{\mathbb{P}^2}(1)$ for any fiber $F \cong \mathbb{P}^2$.

Note that, since X has rational singularities, $h^i(\mathcal{O}_M) = h^i(\mathcal{O}_X) = 0$ for $i = 2, 3$. Hence

$$\chi(\mathcal{O}_S) - h = \chi(\mathcal{O}_M) = 1 - q, \tag{18}$$

i.e.,

$$h = h^0(K_S). \tag{19}$$

Note also that, since $K_X + t\mathcal{K} \approx (\varphi_*(K_M + t\mathcal{K}_M))^{**}$ for $t \geq 1$, from Lemma (1.6) one has an inclusion

$$H^0(M, K_M + t\mathcal{K}_M) \subset H^0(X, K_X + t\mathcal{K}), \quad t \geq 1. \tag{20}$$

Then, since $K_X + t\mathcal{K}$ is not effective for $t = 1, 2$ in case iv), we conclude that $h^0(K_M + \mathcal{K}_M) = h^0(K_M + 2\mathcal{K}_M) = 0$. Thus (15) and (16) give respectively $4\chi(\mathcal{O}_S) + d_2 = d_3 + 6h$ and $6\chi(\mathcal{O}_S) + 3d_2 = 5d_3 + 10h$. By subtracting we obtain $\chi(\mathcal{O}_S) + d_2 = 2d_3 + 2h$. The last two relations give

$$3\chi(\mathcal{O}_S) + d_3 = 4h \tag{21}$$

as well as

$$10h = d_2 + 7\chi(\mathcal{O}_S). \tag{22}$$

Lemma 6.1. *Notation as at the beginning of this section. Assume that X is a $\mathbb{P}^2$-bundle, $p : X \to C$, over a smooth curve C of genus q with $K_X + 3\mathcal{K} \approx p^*H$ for some ample line bundle H on C as in case iv) above. Let $h := h^0(K_M + L)$ and let S be a smooth member of $|L|$. Then we have $d_3 \geq 2$ with equality implying $q = 1$, $\chi(\mathcal{O}_S) = 2$, $h = 2$, $d_2 = 6$, $d \geq 24$, $d_1 \geq 13$.*

Proof. Assume $d_3 = 1$. Then from (19) and (21) we get $h = 4 - 3q$. Thus

$$h^0(K_S) = 4 - 3q. \tag{23}$$

This implies that either $q = 0$ or $q = 1$. If $q = 0$, then $h = 4$. Since $\mathcal{K}$ is nef (indeed ample) on a projective bundle, then it is spanned. Hence $|\mathcal{K}|$ defines a morphism $\sigma : X \to \mathbb{P}^3$. Since $d_3 = d_3' = (\mathcal{K}')^3 = 1$ and $\mathcal{K}$ is ample we conclude that σ is generically one-to-one and finite, and therefore σ is an isomorphism. This contradicts the assumption that X is a $\mathbb{P}^2$-bundle. If $q = 1$, then from (23) we get $h^0(K_S) = \chi(\mathcal{O}_S) = 1$. Thus (21) gives the absurdity that $h = 1$ (see [6, (1.2)]).

If $d_3 = 2$, then we have from (19) and (21),

$$h^0(K_S) = 5 - 3q. \tag{24}$$

This implies that either $q = 0$ or $q = 1$. If $q = 0$, then $h = h^0(K_S) = 5$. As above, since $\mathcal{K}$ is nef on a projective bundle, then it is spanned. Hence $|\mathcal{K}|$ defines a morphism $\sigma : X \to \mathbb{P}^4$. Since $d_3 = d_3' = (\mathcal{K}')^3 = 2$, the image of σ is a quadric in $\mathbb{P}^4$ and σ is generically one-to-one. Since $\mathcal{K}$ is ample, we conclude that σ is finite and hence X is isomorphic to a quadric in $\mathbb{P}^4$. Again this contradicts the fact that X is a $\mathbb{P}^2$-bundle.

If $q = 1$, then by (24), $h^0(K_S) = \chi(\mathcal{O}_S) = 2$. Thus (19) and (22) lead to $h = 2$ and $d_2 = 6$. Since $(\mathcal{M}, \mathcal{L})$ is of log-general type we have $d \geq 8$ by Lemma (1.5). Then $d_1^2 \geq dd_2 \geq 48$ gives $d_1 \geq 7$. If $d_1 = 7$, then $d = 8$ by the above inequality, contradicting the parity condition. By combining the double point formula [8, (1.7)] with Hodge index inequalities we obtain $d \geq 24$, $d_1 \geq 13$. Q.E.D.

As a consequence of the results above, we can describe the case $d_3 = 1$. The cases, when d_3 is small, e.g., $d_3 = 1$, are of special interest because these are among the most extreme cases of log-general type threefolds. One numerical manifestation of this is the fact that as d_3 increases, the Hodge index inequalities force all the other invariants to grow significantly.

Theorem 6.2. *Let $\mathcal{M}$ be a smooth connected 3-fold polarized by a very ample line bundle $\mathcal{L}$. Assume that $\kappa(K_{\mathcal{M}} + \mathcal{L}) = 3$. Let $\pi : (\mathcal{M}, \mathcal{L}) \to (M, L)$, $\varphi : (M, L) \to (X, \mathcal{D})$, $\mathcal{K} \approx K_X + \mathcal{D}$, be the first and the second reduction of $(\mathcal{M}, \mathcal{L})$. Let $h := h^0(K_M + L)$ and let S be a smooth member of $|L|$. If $d_3 = 1$, then either $(X, \mathcal{D}) \cong (\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(5))$ or X is a Gorenstein Del Pezzo threefold with $K_X \approx -2\mathcal{K}$, $h = 3$, $\chi(\mathcal{O}_S) = 4$, $d \geq 9$, $d_1 \geq 6$, $d_2 = 3$.*

Proof. If $K_X + 2\mathcal{K}$ is not nef, we are in one of the special cases i)—iv). In case i) we are done, cases ii) and iii) are not possible since we are assuming $d_3 = 1$, and case iv) as well is not possible by Lemma (6.1).

Thus we can assume that $K_X + 2\mathcal{K}$ is nef, so that $K_X + 3\mathcal{K} = 4\mathcal{K} - \mathcal{D}$ is ample. Therefore $(4\mathcal{K} - \mathcal{D}) \cdot \mathcal{K} \cdot \mathcal{K} > 0$, i.e., $4d_3' - d_2' > 0$ and hence $d_2' = d_2 \leq 3$.

If $d_2 = 1$, we have, by the Hodge index relations, $d_1 = d = 1$ and hence $(\mathcal{M}, \mathcal{L}) \cong (\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1))$, contradicting the log-general type assumption. Since $d_2 \neq 2$ by parity, we conclude that $d_2 = 3$. Also, the log-general type assumption gives $d \geq 8$ (see Lemma (1.5)) and therefore $d_1^2 \geq dd_2 \geq 24$ yields $d_1 \geq 5$. But $d_1 = 5$ implies $d = 8$, which again contradicts parity. Then $d_1 \geq 6$.

Assume $K_X + \mathcal{K} \approx 2\mathcal{K} - \mathcal{D}$ effective. Then $(2\mathcal{K} - \mathcal{D}) \cdot \mathcal{K} \cdot \mathcal{K} \geq 0$ gives $2d_3' \geq d_2'$, or $2d_3 \geq d_2$. This contradicts $d_3 = 1$, $d_2 = 3$. Thus we can assume $h^0(K_X + \mathcal{K}) = 0$ and therefore (15) and (20) yield $4\chi(\mathcal{O}_S) + d_2 = d_3 + 6h$, or $2\chi(\mathcal{O}_S) + 1 = 3h$. Since $h = h^0(K_S)$ by (19), this gives $3 = h + 2q$. Since $h \geq 2$ we conclude that

$$q = 0, \quad h = 3, \quad \text{and} \quad \chi(\mathcal{O}_S) = 4.$$

Then using the double point formula [8, (1.7)] we have

$$d(d-10) \geq 2d_2 + 5d_1 - 12\chi(\mathcal{O}_S) \geq -12,$$

and hence $d \geq 9$. From $d_1^2 \geq dd_2 \geq 27$ we obtain $d_1 \geq 6$.

Assume now $h^0(K_X + 2\mathcal{K}) \geq 1$. Note that $K_X + 2\mathcal{K}$ is a rank 1 reflexive sheaf (since K_X is a reflexive sheaf and $\mathcal{K}$ is a line bundle). Then a section $s \in H^0(X, K_X + 2\mathcal{K})$ vanishes on a Cartier divisor. But the ampleness of $\mathcal{K}$ and the equality

$$(K_X + 2\mathcal{K}) \cdot \mathcal{K} \cdot \mathcal{K} = (3\mathcal{K} - \mathcal{D}) \cdot \mathcal{K} \cdot \mathcal{K} = 3d_3 - d_2 = 0$$

imply that s vanishes on a subvariety $Y \subset X$ of codimension ≥ 2. This gives a contradiction, unless $K_X + 2\mathcal{K} \approx \mathcal{O}_X$ (and $h^0(K_X + 2\mathcal{K}) = 1$). Thus X is a Gorenstein Del Pezzo threefold with $K_X \approx -2\mathcal{K}$.

To conclude the proof it remains to show that $h^0(K_X + 2\mathcal{K}) \geq 1$. In view of (20) it suffices to show that $h^0(K_M + 2\mathcal{K}_M) \geq 1$, where $\mathcal{K}_M = K_M + L$. If $\chi(-2\mathcal{K}_M) \neq 0$, we have, since $\mathcal{K}_M$ is nef and big,

$$\chi(-2\mathcal{K}_M) = -\chi(K_M + 2\mathcal{K}_M) = -h^0(K_M + 2\mathcal{K}_M) \neq 0,$$

so we are done. Thus we can assume $\chi(-2\mathcal{K}_M) = 0$, i.e., $h^0(K_M + 2\mathcal{K}_M) = 0$ and hence also

$$h^0(K_M + \mathcal{K}_M) = \chi(K_M + \mathcal{K}_M) = -\chi(-\mathcal{K}_M) = 0,$$

i.e., $\chi(-\mathcal{K}_M) = 0$. Moreover $\chi(\mathcal{K}_M) = \chi(K_M + L) = h^0(K_M + L) = h = 3$. Therefore from the exact sequence

$$0 \to -L \to \mathcal{O}_M \to \mathcal{O}_S \to 0$$

we obtain $\chi(\mathcal{O}_M) = \chi(\mathcal{O}_S) + \chi(-L) = 4 - \chi(K_M + L) = 1$. Now a standard polynomial argument, by using $\chi(\mathcal{K}_M) = 3$, $\chi(\mathcal{O}_M) = 1$, $\chi(-\mathcal{K}_M) = 0$ and $\chi(-2\mathcal{K}_M) = 0$ and the bigness of $\mathcal{K}_M$ leads to a numerical contradiction. Q.E.D.

Remark 6.3. Notation as in (6.2). Note that the case of X being a Gorenstein Del Pezzo threefold with $K_X \approx -2\mathcal{K}$, of invariants $h = 3$, $\chi(\mathcal{O}_S) = 4$, $d \geq 9$, $d_1 \geq 6$, $d_2 = 3$ is possible. The example in [12] has $\mathcal{L} = 3H$ very ample with $K_{\mathcal{M}} \approx -2H$ and degree $\mathcal{L}^3 = 27$. Thus all reductions are isomorphic with $\mathcal{K} = H$ having $h^0(\mathcal{K}) = 3$, $d_3 = \mathcal{K}^3 = 1$. It would be interesting to have a complete classification of the Gorenstein Del Pezzo threefolds with $K_X \approx -2\mathcal{K}$, and the invariants $h = 3$, $\chi(\mathcal{O}_S) = 4$, $d \geq 9$, $d_1 \geq 6$, $d_2 = 3$. Note that since $K_M + 2\mathcal{K}_M$ is $\mathbb{Q}$-effective, $d_1 \leq 9$ and $d \leq 27$.

7. The case $n = 3$, II

Let $\mathcal{M}$ be a smooth connected 3-fold polarized by a very ample line bundle $\mathcal{L}$. Assume that $\kappa(K_{\mathcal{M}} + \mathcal{L}) \geq 0$. Let $\pi : (\mathcal{M}, \mathcal{L}) \to (M, L)$, $\varphi : (M, L) \to (X, \mathcal{D})$, $\mathcal{K} \approx K_X + \mathcal{D}$, be the first and the second reduction of $(\mathcal{M}, \mathcal{L})$ as in (1.1). Let d_j, $j = 0, 1, 2, 3$, $d_0 = d$, be the pluridegrees of (M, L). Let $h := h^0(K_M + L)$ and let S be a general member of $|L|$. Set also $\mathcal{K}_M := K_M + L$.

In this section we study the case $d_3 = 2$. We will work under the assumption that $K_X + 2\mathcal{K}$ is nef. If this is not the case, in view of the exceptions listed in §6, we know that either $(X, \mathcal{D}) \cong (\mathcal{Q}, \mathcal{O}_{\mathcal{Q}}(1))$, $\mathcal{Q}$ smooth quadric in $\mathbb{P}^4$, or X is as in Lemma (6.1) (see also Remark (6.3)).

Since $K_X + 2\mathcal{K}$ is nef we have $(K_X + 2\mathcal{K}) \cdot \mathcal{K} \cdot \mathcal{K} = (3\mathcal{K} - \mathcal{D}) \cdot \mathcal{K} \cdot \mathcal{K} \geq 0$ and hence

$$3d_3 \geq d_2.$$

Therefore, by parity, $d_2 = 2, 4, 6$. If $d_2 = 2$, then $d_2^2 \geq d_1 d_3 = 2d_1$ gives $d_1 = 2$ and hence $d = 2$. Then we would be in the special case of the quadric in $\mathbb{P}^4$, contradicting the nefness assumption.

Thus either $d_2 = 6$ or $d_2 = 4$. We have the following result, whose proof is given in the two paragraphs below, where the cases $d_2 = 6$ and $d_2 = 4$ are treated separately. Recall that $d \geq 8$ by Lemma (1.5).

Theorem 7.1. *Let $\mathcal{M}$ be a smooth connected 3-fold polarized by a very ample line bundle $\mathcal{L}$. Assume that $\kappa(K_{\mathcal{M}} + \frac{2}{3}\mathcal{L}) \geq 0$. Let $\pi : (\mathcal{M}, \mathcal{L}) \to (M, L)$ be the first reduction of $(\mathcal{M}, \mathcal{L})$. Let d_j, $j = 0, 1, 2, 3$, $d_0 = d$, be the pluridegrees of (M, L). Let $h := h^0(K_M + L)$ and let S be a general member of $|L|$. Assume that $d_3 = 2$. Then either*

1. $\chi(\mathcal{O}_S) = 5$, $h = 4$, $d_2 = 6$, $d_1 \geq 9$, $d \geq 11$, *or*
2. $\chi(\mathcal{O}_S) = 4$, $h = 3$, $d_2 = 4$ *and* $(d, d_1) = (11, 7), (12, 8), (14, 8), (16, 8)$.

7.2. **The case** $d_2 = 6$**.** Let $\varphi : (M, L) \to (X, \mathcal{D})$, $\mathcal{K} \approx K_X + \mathcal{D}$ be the second reduction of $(\mathcal{M}, \mathcal{L})$. First note that $K_{\mathcal{M}} + \frac{2}{3}\mathcal{L} \approx \pi^*\varphi^*(K_X + \frac{2}{3}\mathcal{D}) + Z$, for some effective $\mathbb{Q}$-Cartier divisor Z on $\mathcal{M}$. We also have $K_X + 2\mathcal{K} \approx 3(K_X + \frac{2}{3}\mathcal{D})$. Therefore $\kappa(K_{\mathcal{M}} + \frac{2}{3}\mathcal{L}) = \kappa(K_X + \frac{2}{3}\mathcal{D}) = \kappa(K_X + 2\mathcal{K}) \geq 0$. Thus by Proposition (1.3) we know that $K_X + 2\mathcal{K}$ is nef.

The nefness of $K_X + 2\mathcal{K}$ and the ampleness of $\mathcal{K}$ imply nefness and bigness of $2(K_X + 2\mathcal{K}) - K_X$. Thus, by the Kawamata-Shokurov basepoint free theorem, we conclude that $N(2(K_X + 2\mathcal{K}))$ has a section for $N \gg 0$, and hence $N(K_X + 2\mathcal{K})$ has a section for $N \gg 0$, i.e., $K_X + 2\mathcal{K}$ is $\mathbb{Q}$-effective. From

$$(K_X + 2\mathcal{K}) \cdot \mathcal{K}^2 = (3\mathcal{K} - \mathcal{D}) \cdot \mathcal{K}^2 = 3d_3' - d_2' = 3d_3 - d_2 = 0$$

it thus follows that $N(K_X + 2\mathcal{K}) \approx \mathcal{O}_X$ and therefore $-K_X$ is ample. This implies $q := q(S)(= q(X)) = 0$. From $d_1^2 \geq dd_2 \geq 48$ we get $d_1 \geq 7$. If $d_1 = 7$, then by parity $d \geq 9$, so that $d_1^2 \geq dd_2$ yields to a contradiction. Therefore

$$d_1 \geq 8. \tag{25}$$

From $(K_X + \mathcal{K}) \cdot \mathcal{K}^2 = (2\mathcal{K} - \mathcal{D}) \cdot \mathcal{K}^2 = 2d_3' - d_2' = 2d_3 - d_2 < 0$, we conclude that $h^0(K_X + \mathcal{K}) = 0$. Recalling (20) we have $h^0(K_M + \mathcal{K}_M) = 0$. Then relation (15) gives $4\chi(\mathcal{O}_S) + d_2 = d_3 + 6h$, or

$$2\chi(\mathcal{O}_S) + 2 = 3h. \tag{26}$$

Noether's inequality gives $d_2 \geq 2\chi(\mathcal{O}_S) - 6$, so that $\chi(\mathcal{O}_S) \leq 6$ and hence (26) yields either $\chi(\mathcal{O}_S) = h = 2$, or $\chi(\mathcal{O}_S) = 5$, $h = 4$.

Note that the case $\chi(\mathcal{O}_S) = h = 2$ is not possible. Indeed from $\chi(\mathcal{O}_S) = 2$ and $q = 0$ we have $h^0(K_S) = 1$. Then we see that $h = h^0(K_M + L) = 2$ is not possible by considering the cohomology sequence associated to the exact sequence

$$0 \to K_M \to K_M + L \to K_S \to 0.$$

Therefore $\chi(\mathcal{O}_S) = 5$, $h = 4$. Since $q = 0$ one has $h^0(K_S) = 4$. From the double point formula [8, (1.7)] we have $d(d - 10) + 48 \geq 5d_1$. Recalling (25) we obtain $d \geq 10$.

Assume that $|\mathcal{L}|$ embeds $\mathcal{M}$ in $\mathbb{P}^N$ with $N \geq 6$. Then $d \geq 11$. Indeed, if $d = 10$, Castelnuovo's bound gives $g(\mathcal{L}) \leq \text{Castel}(10, 4) = 9$, and hence we find the contradiction $18 \leq d + d_1 = \widehat{d} + \widehat{d_1} = 2g(\mathcal{L}) - 2 \leq 16$.

Assume now that $|\mathcal{L}|$ embeds $\mathcal{M}$ in $\mathbb{P}^5$. Since $d \geq 10$, [2, (4.4.1)] applies to say that $\mathcal{M} \cong M$. Thus the double point formula [8, (1.7)] for $d = 10$ gives $12\chi(\mathcal{O}_S) = 2d_2 + 5d_1$, whence the numerical contradiction $60 = 12 + 5d_1$. Therefore we conclude that $d \geq 11$.

From $d_1^2 \geq dd_2 \geq 66$ we also get $d_1 \geq 9$. Summarizing, we have the numerical bounds $d_3 = 2$, $d_2 = 6$, $d_1 \geq 9$, $d \geq 11$.

Example 7.3. Let us construct an explicit example satisfying the numerical bounds above. Let $p : \mathcal{M} \to \mathbb{P}^3$ be a finite double cover of $\mathbb{P}^3$ branched along a smooth quartic surface B. Then $K_{\mathcal{M}} \approx p^*\mathcal{O}_{\mathbb{P}^3}(-2)$. Let $\mathcal{L} := p^*\mathcal{O}_{\mathbb{P}^3}(3)$, so that $K_{\mathcal{M}} + \mathcal{L} \approx p^*\mathcal{O}_{\mathbb{P}^3}(1)$. Note that $\mathcal{L}$ and $K_{\mathcal{M}} + 2\mathcal{L} \approx p^*\mathcal{O}_{\mathbb{P}^3}(4)$ are very ample (see e.g., [12]). Hence in particular $(\mathcal{M}, \mathcal{L})$ is isomorphic to its own first reduction (M, L). The invariants are $d = 2\mathcal{O}_{\mathbb{P}^3}(3)^3 = 54$, $d_1 = 2\mathcal{O}_{\mathbb{P}^3}(1) \cdot \mathcal{O}_{\mathbb{P}^3}(3)^2 = 18$, $d_2 = 2\mathcal{O}_{\mathbb{P}^3}(1)^2 \cdot \mathcal{O}_{\mathbb{P}^3}(3) = 6$, $d_3 = 2\mathcal{O}_{\mathbb{P}^3}(1)^3 = 2$. Moreover $h = h^0(K_{\mathcal{M}} + \mathcal{L}) = h^0(p^*\mathcal{O}_{\mathbb{P}^3}(1)) = 4$. To see this, consider the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-3) \to \mathcal{O}_{\mathbb{P}^3}(1) \to \mathcal{O}_{\mathbb{P}^3}(1)_{|B} \to 0.$$

Then $h^0(\mathcal{O}_{\mathbb{P}^3}(1)_{|B}) = h^0(\mathcal{O}_{\mathbb{P}^3}(1)) = 4$. On $\mathcal{M}$ we have the exact sequence (identifying B with $p^{-1}(B)$)

$$0 \to p^*\mathcal{O}_{\mathbb{P}^3}(-2) \to \mathcal{O}_{\mathcal{M}} \to \mathcal{O}_B \to 0.$$

Tensoring with $K_{\mathcal{M}} + \mathcal{L}$ and computing the cohomology we obtain

$$h = h^0(K_{\mathcal{M}} + \mathcal{L}) = h^0((K_{\mathcal{M}} + \mathcal{L})_{|B}) = h^0(\mathcal{O}_{\mathbb{P}^3}(1)_{|B}) = 4.$$

Note also that $K_{\mathcal{M}} + \mathcal{L}$ is ample and spanned and gives a two-to-one map (see [12]). It would be interesting to have a complete classification of the threefolds as in Theorem (7.1) with the invariants $\chi(\mathcal{O}_S) = 5$, $h = 4$, $d_2 = 6$, $d_1 \geq 9$, $d \geq 11$. Note that from $\kappa(K_{\mathcal{M}} + \frac{2}{3}\mathcal{L}) = 0$, it follows that $d_1 \leq 18$ and $d \leq 54$.

7.4. **The case $d_2 = 4$.** Let us first prove the following claim.

Claim. $h^0(K_X + \mathcal{K}) = 0$.

Proof. Assume otherwise. Then from $(K_X + \mathcal{K}) \cdot \mathcal{K}^2 = (2\mathcal{K} - \mathcal{D}) \cdot \mathcal{K}^2 = 2d_3' - d_2' = 2d_3 - d_2 = 0$ we see that $2(K_X + \mathcal{K}) \approx \mathcal{O}_X$. Hence we have a section of $2(K_X + \mathcal{K})$ and hence a section of $K_X + \mathcal{K}$ which doesn't vanish at regular points of X. This implies that $K_X + \mathcal{K} \approx \mathcal{O}_X$.

Thus X is Gorenstein, $q(X) = 0$, $h^0(K_X + \mathcal{K}) = 1$, $h^2(\mathcal{O}_X) = h^3(\mathcal{O}_X) = 0$ by Kodaira vanishing and $\chi(\mathcal{O}_X) = 1$. Since X is Gorenstein we have $K_M + \mathcal{K}_M \approx \varphi^*(K_X + \mathcal{K}) + J$, where J is an integral divisor. Therefore J is a line bundle since M is smooth. Recalling that $h^0(K_M + \mathcal{K}_M) \leq h^0(K_X + \mathcal{K}) = h^0(\varphi_*(K_M + \mathcal{K}_M)^{**})$ by Lemma (1.6), we conclude from $h^0(K_X + \mathcal{K}) = 1$ that $h^0(K_M + \mathcal{K}_M) = 1$. Thus relation (15) yields

$$2 = 2h^0(K_M + \mathcal{K}_M) = d_3 + 6h - d_2 - 4\chi(\mathcal{O}_S).$$

Since $\chi(\mathcal{O}_S) = h + 1$ from (18), and $d_3 = 2$, $d_2 = 4$, we find $3h = 2(\chi(\mathcal{O}_S) + 1)$. Noether's formula gives $4 = d_2 \geq 2\chi(\mathcal{O}_S) - 6$, i.e., $\chi(\mathcal{O}_S) \leq 5$. Recalling that $h \geq 2$ we infer that either $h = 4$, $\chi(\mathcal{O}_S) = 5$, or $h = \chi(\mathcal{O}_S) = 2$. Exactly the same argument as in paragraph (7.2) rules out the case $h = \chi(\mathcal{O}_S) = 2$. Then

$d_2 = 2\chi(\mathcal{O}_S) - 6$, so that S is a Horikawa surface. Then from [11, Lemma (1.1)] we know that K_S is spanned since $d_2 = K_S^2$ is even. Thus from the exact sequence

$$0 \to K_M \to K_M + L \to K_S \to 0,$$

by using $h^2(\mathcal{O}_M) = h^1(K_M) = 0$, we infer that $\mathcal{K}_M = K_M + L$ is spanned. Let $\psi : M \to \mathbb{P}^{h-1} = \mathbb{P}^3$ be the surjective map associated to $|\mathcal{K}_M|$. Since $(K_M + L)^3 = d_3 = 2$, the morphism ψ is generically two-to-one. Since $\mathcal{K}_M \approx \varphi^*\mathcal{K}$, ψ factors through the second reduction map φ, $\psi = f \circ \varphi$, $f : X \to \mathbb{P}^3$, and f is finite since $\mathcal{K}$ is ample. We have $\mathcal{K} \approx f^*\mathcal{O}_{\mathbb{P}^3}(1)$ and $h = h^0(\mathcal{K}_M) = h^0(\mathcal{K}) = 4$. Let B be the branch locus of ψ. Then $\mathcal{O}_{\mathbb{P}^3}(B) \approx \mathcal{O}_{\mathbb{P}^3}(2b)$, $2b := \deg B$. One has $L \approx \psi^*\mathcal{O}_{\mathbb{P}^3}(a)$ for some integer a, $K_M \approx \psi^*\mathcal{O}_{\mathbb{P}^3}(b-4)$ and hence $\psi^*\mathcal{O}_{\mathbb{P}^3}(a+b-4) \approx \mathcal{K}_M \approx \psi^*\mathcal{O}_{\mathbb{P}^3}(1)$. Therefore $a + b = 5$. From $4 = d_2 = \mathcal{K}_M^2 \cdot L = 2\mathcal{O}_{\mathbb{P}^3}(1)^2 \cdot \mathcal{O}_{\mathbb{P}^3}(5-b) = 2(5-b)$, we see that $b = 3$.

To conclude the proof of the claim it remains to show that the case $b = 3$ does not occur. To see this consider the exact sequences (identifying B with $\psi^{-1}(B)$)

$$0 \to \mathcal{O}_{\mathbb{P}^3}(5-3b) \to \mathcal{O}_{\mathbb{P}^3}(5-b) \to \mathcal{O}_{\mathbb{P}^3}(5-b)_{|B} \to 0$$

and

$$0 \to L - B \approx \psi^*\mathcal{O}_{\mathbb{P}^3}(5-2b) \to L \to L_B \to 0.$$

If $b = 3$ then $5 - 2b < 0$, so that

$$h^0(\mathcal{O}_{\mathbb{P}^3}(5-b)) = h^0(\mathcal{O}_{\mathbb{P}^3}(5-b)_{|B}) = h^0(L_B) \geq h^0(\mathcal{O}_{\mathbb{P}^3}(5-b)).$$

Thus $h^0(L) = h^0(\mathcal{O}_{\mathbb{P}^3}(5-b))$, i.e., all sections of L pullback from $\mathbb{P}^3$. Therefore $|L|$ doesn't give a birational map. This contradicts the fact that L is very ample outside of a finite set of points. $\square$

Recalling Lemma (1.6), from now on we can thus assume that $h^0(K_X + \mathcal{K}) = h^0(K_M + \mathcal{K}_M) = 0$. Then since $d_3 = 2$, $d_2 = 4$, from relation (15) we find $2\chi(\mathcal{O}_S) + 1 = 3h$. Noether's formula $d_2 \geq 2\chi(\mathcal{O}_S) - 6$ gives again $\chi(\mathcal{O}_S) \leq 5$. Thus, recalling that $h \geq 2$, the above equality $2\chi(\mathcal{O}_S) + 1 = 3h$ gives $\chi(\mathcal{O}_S) = 4$, $h = 3$. From $d_3 d_1 = 2d_1 \leq d_2^2 = 16$ we have $d_1 \leq 8$. Then $dd_2 = 4d \leq d_1^2 \leq 64$ yields $d \leq 16$. From [8, (1.7)] one has $d(d-10) + 40 \geq 5d_1$. Since $d_1 \geq 6$, this implies $d \geq 9$. If $d = 9$, then $d_1 \geq 7$ by parity and hence the inequality above gives again a contradiction. Therefore $d \geq 10$. Then $d_1^2 \geq dd_2 \geq 40$ yields $d_1 \geq 7$. Thus we have

$$d_3 = 2, \quad d_2 = 4, \quad 8 \geq d_1 \geq 7, \quad 16 \geq d \geq 10. \tag{27}$$

Assume $d = 10$. Then $\widehat{d} \leq 10$ and $d_1 = 8$ by parity. Suppose that $|\mathcal{L}|$ embeds $\mathcal{M}$ in $\mathbb{P}^N$ with $N \geq 6$. Then $g(\mathcal{L}) \leq \mathrm{Castel}(10, 4) = 9$. This gives the numerical contradiction $16 \geq 2g(\mathcal{L}) - 2 = \widehat{d} + \widehat{d}_1 = d + d_1 \geq 18$.

Thus if $d = 10$, we see that $|\mathcal{L}|$ embeds $\mathcal{M}$ in $\mathbb{P}^5$. By combining Lemma (1.5) and [2, (4.4.1)] we conclude that $(\mathcal{M}, \mathcal{L}) \cong (M, L)$, $d = \widehat{d} = 10$, $d_1 = \widehat{d}_1 = 8$. Since there are no 3-folds in $\mathbb{P}^5$ of degree 10 with $d_1 = 8$ (see [3, Chapter 6]), we conclude that $d \geq 11$.

Recalling (27) and by using parity, we have the following possibilities for d, d_1. Precisely, $(d, d_1) = (11, 7), (12, 8), (14, 8), (16, 8)$. Note that if $d = 13$, then $d_1 = 7$ by parity and $d_1^2 \geq dd_2 \geq 52$ gives a contradiction. Similarly, if $d = 15$, then $d_1 = 7$, and we contradict again $d_1^2 \geq dd_2$.

8. A lower bound for the number of sections of the second adjoint bundle

Let $\mathcal{M}$ be a smooth connected n-fold polarized by a very ample line bundle $\mathcal{L}$, $n \geq 3$. Assume that $\kappa(K_{\mathcal{M}} + (n-2)\mathcal{L}) = n$. Let $\pi : (\mathcal{M}, \mathcal{L}) \to (M, L)$, $\varphi : (M, L) \to (X, \mathcal{D})$, $\mathcal{K} \approx K_X + (n-2)\mathcal{D}$, be the first and the second reduction of $(\mathcal{M}, \mathcal{L})$ as in (1.1). Let $h := h^0(K_{\mathcal{M}} + (n-2)\mathcal{L})$. In [6, (1.2), (3.3)] we showed that $h \geq 2$ under the assumption $\kappa(K_{\mathcal{M}} + (n-2)\mathcal{L}) = n$ and moreover, if $\kappa(\mathcal{M}) \geq 0$, that $h \geq 5$ with equality only if $n = 3$ and $(\mathcal{M}, \mathcal{L})$ is a degree 5 hypersurface in $\mathbb{P}^4$. In this section we prove that for $n \geq 5$, under the further assumption that $K_X + (n-3)\mathcal{K}$ is nef and big (see §2) and by using the pluridegrees bounds stated in the previous sections, one has the general lower bound $h \geq 5$. Let $\mathcal{V}$ be the smooth 5-fold section obtained as the transversal intersection of $n-5$ general members of $|\mathcal{L}|$ and let $\mathcal{L}_{\mathcal{V}}$ be the restriction of $\mathcal{L}$ to $\mathcal{V}$. The Kodaira vanishing theorem yields $h^0(K_{\mathcal{M}} + (n-2)\mathcal{L}) \geq h^0(K_{\mathcal{V}} + 3\mathcal{L}_{\mathcal{V}})$. Thus, since we have better bounds as soon as n increases, we are reduced to proving the following result.

Proposition 8.1. *Let $\mathcal{M}$ be a smooth connected 5-fold polarized by a very ample line bundle $\mathcal{L}$. Assume that $\kappa(K_{\mathcal{M}} + 2\mathcal{L}) = 5$. Let $\pi : (\mathcal{M}, \mathcal{L}) \to (M, L)$ be the first reduction of $(\mathcal{M}, \mathcal{L})$. Then $h := h^0(K_M + 3L) = h^0(K_{\mathcal{M}} + 3\mathcal{L}) \geq 5$.*

Proof. Let $\varphi : (M, L) \to (X, \mathcal{D})$, $\mathcal{K} \approx K_X + 3\mathcal{D}$, be the second reduction of $(\mathcal{M}, \mathcal{L})$. Let d_j's, $j = 0, 1, 2, 3, 4, 5$, $d_0 = d$, be the pluridegrees of (M, L). Note that $K_X + 2\mathcal{K} \approx 3(K_X + 2\mathcal{D})$ is nef and big by (1.2).

Let S be the smooth surface obtained as the transversal intersection of $n - 2 = 3$ general members of $|L|$. Then S is a minimal surface of general type. Since $d_2 \geq 24$ by (4.1), from Miyaoka's inequality we infer that $\chi(\mathcal{O}_S) \geq 3$.

Assume $\chi(\mathcal{O}_S) = 3$. Then since $d_1 \geq 19$, $d_2 \geq 24$ from (4.1), the double point formula [8, (1.7)] gives $d(d-10) \geq 107$, and hence $d \geq 17$. Therefore (10) gives

$$d \geq 17, \quad d_1 \geq 21, \quad d_2 \geq 25, \quad d_3 \geq 30, \quad d_4 \geq 36, \quad d_5 \geq 43.$$

If $d_2 = 25$, then $d_3 \geq 31$ by parity, so that we find the contradiction $d_2^2 = 625 \geq d_1 d_3 \geq 651$. Therefore $d_2 \geq 26$ and hence $d_1^2 \geq d d_2 \geq 442$ yields $d_1 \geq 22$. Thus [8, (1.7)] gives $d(d-10) \geq 126$, or $d \geq 18$. If $d_2 = 26$, then $d_3 \geq 31$ by (10) and hence $d_3 \geq 32$ by parity, so that $d_2^2 \geq d_1 d_3 \geq 704$ gives $d_2 \geq 27$. Then $d_1^2 \geq d d_2 = 486$ yields $d_1 \geq 23$, as well as $d_3 \geq 32$, $d_4 \geq 37$, $d_5 \geq 44$ by (10). Summarizing, we have

$$d \geq 18, \quad d_1 \geq 23, \quad d_2 \geq 27, \quad d_3 \geq 32, \quad d_4 \geq 37, \quad d_5 \geq 44.$$

From $d_2^2 \geq d_1 d_3 \geq 736$ we get $d_2 \geq 28$. Thus by (10) we obtain $d_3 \geq 34$, $d_4 \geq 41$, $d_5 \geq 49$ and therefore $d_3^2 \geq d_2 d_4 \geq 1184$ yields $d_3 \geq 35$. Again, (10) gives $d_4 \geq 42$, $d_5 \geq 50$. From $d_2^2 \geq d_1 d_3 \geq 805$ we find $d_2 \geq 29$. If $d_1 = 23$ then $d \geq 19$ by parity, so that $d_2 \leq d_1^2/d$ leads to a contradiction. Thus we have

$$d \geq 18, \quad d_1 \geq 24, \quad d_2 \geq 29, \quad d_3 \geq 35, \quad d_4 \geq 42, \quad d_5 \geq 50. \tag{28}$$

Note that Miyaoka inequality $\chi(\mathcal{O}_S) > d_2/9$ yields $\chi(\mathcal{O}_S) \geq 4$. Let $\mathcal{V}$ be the smooth 3-fold section of $\mathcal{M}$ obtained as transversal intersection of two general members of $|\mathcal{L}|$ and let $\mathcal{L}_{\mathcal{V}}$ be the restriction of $\mathcal{L}$ to $\mathcal{V}$. Let $h' := h^0(K_{\mathcal{V}} + \mathcal{L}_{\mathcal{V}})$. Then by Kodaira vanishing one has $h \geq h'$. Thus it suffices to show that $h' \geq 5$.

By using (28), and recalling that d_j equals the pluridegrees $d_j(\mathcal{V})$ of $(\mathcal{V}, \mathcal{L}_{\mathcal{V}})$, $j = 0, 1, 2, 3$, Tsuji's inequality gives $h' \geq 4$. Clearly we can assume $\chi(\mathcal{O}_S) = 4$

since otherwise Tsuji's inequality gives $h' \geq \frac{5}{2} + 1 + \frac{35}{64}$, i.e., $h' \geq 5$, and we would be done.

Note also that we can assume $h^0(K_{\mathcal{V}}) = 0$, since otherwise a section of $\Gamma(K_{\mathcal{V}})$ gives an embedding $\Gamma(\mathcal{L}_{\mathcal{V}}) \hookrightarrow \Gamma(K_{\mathcal{V}} + \mathcal{L}_{\mathcal{V}})$ and hence $h' := h^0(K_{\mathcal{V}} + \mathcal{L}_{\mathcal{V}}) \geq 5$. Then the double point formula [8, Prop. (1.5)] yields $44h' + 58\chi(\mathcal{O}_S) + 4 \geq 12d_2 + 17d_1 + d_3 + (20 - d)d$, or, with $h' = \chi(\mathcal{O}_S) = 4$, $(d - 20)d \geq 269$. This implies $d \geq 30$ and therefore $d_1 \geq 36$, $d_2 \geq 43$, $d_3 \geq 51$ by (10). Thus Tsuji's inequality gives $h \geq 2 + \frac{36}{24} + \frac{51}{64}$, or $h \geq 5$. Q.E.D.

References

[1] M.C. Beltrametti, M.L. Fania, and A.J. Sommese, "On the adjunction theoretic classification of projective varieties," Math. Ann. 290 (1991), 31–62. MR **92f:**14038

[2] M.C. Beltrametti, M. Schneider, and A.J. Sommese, "Threefolds of degree 11 in $\mathbb{P}^5$," in *Complex Projective Geometry*, ed. by G. Ellingsrud, C. Peskine, G. Sacchiero, and S.A. Stromme, London Math. Soc. Lecture Note Ser. 179 (1992), 59–80. MR **94d:**14037

[3] M.C. Beltrametti, M. Schneider, and A.J. Sommese, *Some special properties of the adjunction theory for* 3*-folds in* $\mathbb{P}^5$, Memoirs Amer. Math. Soc., v. 116, n. 554 (1995). MR **95k:**14055

[4] M.C. Beltrametti and A.J. Sommese, "Special results in adjunction theory in dimension four and five," Ark. Mat. 31 (1993), 197–208. MR **95c:**14008

[5] M.C. Beltrametti and A.J. Sommese, "On the adjunction theoretic classification of polarized varieties," J. Reine Angew. Math. 427 (1992), 157–192. MR **93d:**14012

[6] M.C. Beltrametti and A.J. Sommese, "On the dimension of the adjoint linear system for threefolds," Ann. Scuola Norm. Sup. Pisa Cl. Sci. Ser. (4), 22 (1995), 1–24. MR **96e:**14005

[7] M.C. Beltrametti and A.J. Sommese, *The Adjunction Theory of Complex Projective Varieties*, Expositions in Mathematics, 16, W. de Gruyter, 1995. MR **96f:**14004

[8] M.C. Beltrametti and A.J. Sommese, "On the second adjunction mapping. The case of a 1-dimensional image," Trans. Amer. Math. Soc. 349 (1997), 3277–3302. MR **97j:**14043

[9] M.C. Beltrametti and A.J. Sommese, "On the degree and the birationality of the second adjunction mapping," International Journal of Mathematics, to appear.

[10] T. Fujita, "Remarks on quasi-polarized varieties," Nagoya Math. J. 115 (1989), 105–123. MR **90i:**14045

[11] E. Horikawa, "Algebraic surfaces of general type with small c_1^2, I," Ann. Math. 104 (1976), 357–387. MR **54:**12789

[12] A. Lanteri, M. Palleschi, and A.J. Sommese, "On triple covers of $\mathbb{P}^n$ as very ample divisors," in *Classification of Algebraic Varieties, Proceedings L'Aquila, 1992*, ed. by C. Ciliberto, E.L. Livorni, and A.J. Sommese, Contemp. Math. 162 (1994), 277–292. MR **95f:**14012

[13] E.L. Livorni and A.J. Sommese, "Threefolds of non-negative Kodaira dimension with sectional genus less than or equal to 15," Ann. Scuola Norm. Sup. Pisa Cl. Sci. Ser. (4) 13 (1986), 537–558. MR **88m:**14032

Dipartimento di Matematica, Università Degli Studi di Genova, Via Dodecaneso 35, I-16146 Genova, Italy

E-mail address: `beltrame@dima.unige.it`

Department of Mathematics, University of Notre Dame, Notre Dame, Indiana 46556

E-mail address: `sommese@nd.edu`

URL: `http://www.nd.edu/~sommese`